Abstract

Let $G = (V, E)$ be a bipartite graph with bipartition $X, Y$ and let $|X| \leq |Y|$. A dominator sequence in $G$ is a sequence of vertices $(x_1, x_2, \ldots, x_k)$ in $X$ such that for each $i$ with $2 \leq i \leq k$, the vertex $x_i$ dominates at least one vertex in $Y$ which is not dominated by $x_1, x_2, \ldots, x_{i-1}$. The maximum length of a dominator sequence in $G$ is called the dominator sequence number of $G$ and is denoted by $l(G)$. In this paper we present several basic results on this parameter. We prove that the decision problem for the parameter $l(G)$ is NP-Complete. We obtain bounds for $l(G)$ and discuss applications in the study of optical networks.

Keywords: Bipartite graph, domination number, dominator sequence number, optical networks, survivability of IP-over-WDM networks.

1 Introduction

Throughout this paper $G = (V, E)$ stands for a bipartite graph with bipartition $X, Y$ and we assume that $G$ does not have isolated vertices. For graph theoretic terminology we refer to Chartrand and Lesniak [4].

Hedetniemi and Laskar [6, 7] proposed a bipartite theory of graphs and suggested an equivalent formulation of several concepts on graphs as concepts for bipartite graphs. One among them is the concept of $Y$-domination, where for a connected bipartite graph
\[ G = (V, E) \] with bipartition \( X, Y \), a subset \( D \) of \( X \) is a \( Y \)-dominating set of \( G \) if every \( y \in Y \) is adjacent to at least one vertex of \( D \). The minimum order \( \gamma_Y(G) \) of a \( Y \)-dominating set of \( G \) is called the \( Y \)-domination number of \( G \). In this paper we introduce the concept of dominator sequence number of a bipartite graph, which is motivated by an application in optical networks. We present several basic results on this concept. We establish the NP-Completeness of the problem of finding the dominator sequence number of bipartite graphs and we give a linear time algorithm for constructing a maximum dominator sequence in a tree. Other possible generalizations of the concept of dominator sequence are given in Section 6. We need the following definitions.

**Definition 1.1.** A Cartesian product \( G = G_1 \square G_2 \) has \( V(G) = V(G_1) \times V(G_2) \), and two vertices \((u_1, u_2)\) and \((v_1, v_2)\) of \( G \) are adjacent if and only if either \( u_1 = v_1 \) and \( u_2v_2 \in E(G_2) \) or \( u_2 = v_2 \) and \( u_1v_1 \in E(G_1) \).

**Definition 1.2.** Let \( G \) be a connected graph. The subdivision graph \( S(G) \) of a graph \( G \) is the graph obtained from \( G \) by replacing each edge \( uv \) of \( G \) by a new vertex \( w \) along with edges \( uw \) and \( vw \).

**Definition 1.3.** For a graph \( G = (V, E) \), the shadow graph of \( G \) is the graph \( G_s \) with vertex set \( V \cup V' \) where \( V' = \{ x' : x \in V \} \) and is disjoint from \( V \), and edge set \( E' = E \cup \{ xy' : xy \in E \} \).

**Definition 1.4.** Let \( G \) be a bipartite graph with bipartition \( X, Y \). The bipartite complement \( G^{bc} \) of \( G \) has the same bipartition \( X, Y \) and \( x_1y_1 \in E(G^{bc}) \) if and only if \( x_1y_1 \notin E(G) \).

## 2 Motivation: Survivable Logical Topology Mapping in an IP-over-WDM Optical Network

The concept of layering plays an important role in the design of communication networks and protocols. An IP(Internet Protocol) over WDM(Wavelength Division Multiplexing) network is an example of a layered network. Here the WDM optical network is the physical layer represented by a graph \( G_p \). The IP layer is the logical layer represented by a graph \( G_l \). Without loss of generality we assume that \( G_l \) has the same vertex set as \( G_p \). Also we assume that \( G_l \) and \( G_p \) are both 2-edge connected.

Each edge in \( G_l \) between vertices \( v \) and \( w \) corresponds to a path(called lightpath) between \( v \) and \( w \) in \( G_p[5] \). To transmit information from vertex \( u \) to vertex \( v \), first a \( u-v \) path \( P \) in \( G_l \) is identified. Then the information is transmitted using lightpaths corresponding to the logical links in \( P \). If an edge in \( G_p \) fails, then several edges in \( G_l \) could fail causing \( G_l \) to become disconnected and thereby disrupting transmission of information. Survivable logical topology mapping (SLTM) is to map each edge in \( G_l \) into a lightpath in \( G_p \), such that a single edge failure in \( G_p \) does not disconnect \( G_l \).

The SLTM problem has been studied using two approaches. The approach using mathematical programming formulation was pioneered in [12]. The other approach, called the structural approach, uses graph-theoretic concepts and was pioneered in [9].

The structural approach (though described differently in [9]) can be explained using the concept of ear decomposition of \( G_l \). This approach may be viewed as constructing an ear decomposition of \( G_l \) and mapping the edges in each ear into edge-disjoint lightpaths.
in $G_p$. If no ear decomposition that admits such a mapping is available then the given SLTM problem is infeasible. For details of this approach see [9], [13] and [14].

For purposes of the application under discussion, our definition of an ear given next is slightly more general than the traditional definition of an ear given in [17]. The first ear is a circuit in the given graph. Each subsequent ear is a circuit obtained by contracting the other ears already selected. If an ear has exactly one edge, the corresponding lightpath in the physical graph $G_p$ can be selected arbitrarily and so such ears are not of interest in the cross layer survivability mapping problem. Since circuits/cutsets, and edge contraction/deletion are dual concepts[11], the question arises whether there exists a dual cutset-based approach for the SLTM problem. This question has been discussed in [13] and [14]. A brief outline of the cutset-based approach is as follows.

Consider a connected, undirected and simple graph $G = (V,E)$ with vertex set $V$ and edge set $E$. Let $|V| = n$ and $|E| = m$. Consider a partition $(S, \overline{S})$ of $V$, where $\overline{S} = V - S$. Then the set of edges with one end in $S$ and the other in $\overline{S}$ is called a cut of $G$.

![Figure 1: (a) A graph with a spanning tree (bold lines) (b) A cut.](image)

For example consider the graph $G$ in Fig.1(a). Here the vertices are numbered 1, 2, ..., 6. The bold edges in this figure denote the branches of a spanning tree $T$ of $G$ and the dotted edges are the chords of this tree. The partition $(S, \overline{S})$ with $S = \{1, 4, 6\}$ and $\overline{S} = \{2, 3, 5\}$ defines the cut shown in Fig.1(b). If we remove a branch $b$ from a spanning tree $T$, then the tree $T$ gets disconnected resulting in two trees (not spanning) $T_1$ and $T_2$. The sets of nodes in $T_1$ and $T_2$ define a partition of $V$. The corresponding cut is called the fundamental cutset of $T$ with respect to the branch $b$. For example, if we remove the branch $b_3$ from the tree $T$ of Fig.1(a) then we get trees $T_1$ and $T_2$ given by the branches $\{b_1, b_2, b_3\}$ and $\{b_4\}$, respectively. The corresponding fundamental cutset $Q(b_3)$ consists of the edges $\{b_3, c_1, c_3, c_4, c_5, c_6\}$. Note that the subgraphs induced by the vertex sets of $T_1$ and $T_2$ are both connected.

The fundamental cutset matrix with respect to the tree $T$ is defined as $Q^f = [q_{ij}]_{(n-1) \times m}$. The matrix $Q^f$ has $(n - 1)$ rows, one for each fundamental cutset and one column for each edge. The entry $q_{ij}$ is defined as

$$q_{ij} = \begin{cases} 1, & \text{if } Q(b_i) \text{ contains edge } j \\ 0, & \text{otherwise.} \end{cases}$$
Arranging the rows of $Q^f$ such that the $j^{th}$ row corresponds to $f$-cutset $Q(b_j)$ and the columns correspond to the edges in the order $\{b_1, b_2, \ldots, b_{n-1}, c_1, c_2, \ldots, c_{m-n+1}\}$ the matrix $Q^f$ can be written as $Q^f = [U|Q^{fc}]$. For example, the $Q^f$ matrix with respect to the tree $T$ of Fig.1(a) is given below.

$$
\begin{pmatrix}
  b_1 & b_2 & b_3 & b_4 & b_5 \\
  1 & 0 & 0 & 0 & 0 \\
  b_2 & 0 & 1 & 0 & 0 \\
  b_3 & 0 & 0 & 1 & 0 \\
  b_4 & 0 & 0 & 0 & 1 \\
  b_5 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\rightarrow (1)
$$

An ordered sequence $Q(b_1), Q(b_2), \ldots, Q(b_k)$ is a cutset cover sequence or simply a $Q$-sequence of length $k$ if

a) $[Q(b_j) - b_j - \bigcup_{p=1}^{j-1} Q(b_p)] \neq \emptyset, 2 \leq j \leq k$.

b) $\bigcup_{p=1}^{k} Q(b_p) = E - \{\text{branches not in the } Q\text{-sequence}\}$.

Note that for a given spanning tree and its $f$-cutsets, there may be more than one $Q$-sequence. For example, for the fundamental cutsets given in (1), following are the three $Q$-sequences.

1. $Q(b_1), Q(b_3), Q(b_2), Q(b_2)$.
2. $Q(b_4), Q(b_5), Q(b_1), Q(b_2)$.
3. $Q(b_1), Q(b_2), Q(b_4)$.

Without loss of generality assume that $Q(b_1), Q(b_2), \ldots, Q(b_k)$ is a $Q$-sequence of length $k$. Let us define $\hat{S}(b_j)$ as follows:

a) $\hat{S}(b_1) = Q(b_1) - b_1$.

b) $\hat{S}(b_j) = Q(b_j) - b_j - \bigcup_{p=1}^{j-1} Q(b_p), 2 \leq j \leq k$.

The sets $b_i \cup \hat{S}(b_i)$ are called cutset-ears. The ears corresponding to the branches in a cutset cover sequence $D$ form a cutset-ear decomposition of the graph obtained from $G_l$ by contracting the branches that are not in the cutset cover sequence.

The cutset-based([13],[14]) method has the following steps:

1. Map the edges in each cutset-ear $b_i \cup \hat{S}(b_i)$ into mutually edge-disjoint paths in $G_p$.
2. For each branch $b_i$ not in the cutset-cover sequence, add a parallel edge in $G_l$ and map these edges into mutually edge-disjoint paths in $G_p$. The newly added edges are called protection edges.
Each protection edge entails the provision of additional resources in \( G_p \) and so is expensive. So, to minimize this cost, we need to start with a cutset-cover sequence which has the largest number of branches possible.

We now construct a bipartite graph \( G(X,Y) \) where each vertex in \( X \) corresponds to a branch \( b_i \) of a spanning tree \( T \); each vertex in \( Y \) corresponds to a chord \( c_i \) and \( b_i \) is joined to \( c_j \) if and only if the fundamental cutset with respect to \( b_i \) contains the chord \( c_j \). Then a cutset-cover sequence in \( G \) corresponds to a dominator sequence, which we formally define in the next section. The SLTM problem requires the construction of a longest dominator sequence.

### 3 Basic Results

In this section we introduce the concept of a dominator sequence in a bipartite graph and present several basic results.

**Definition 3.1.** Let \( G \) be a bipartite graph with bipartition \( X,Y \). A Y-dominator sequence is a sequence of vertices \( (x_1,x_2,\ldots,x_k) \) in \( X \) such that for each \( i, 1 \leq i \leq k \), there exists \( y_i \in Y \) such that \( y_i \) is dominated by \( x_i \) and \( y_i \) is not dominated by any of the vertices \( x_1,x_2,\ldots,x_{i-1} \), or equivalently, \( y_i \in N(x_i)-(N(x_1)\cup N(x_2)\cup \cdots \cup N(x_{i-1})) \). The maximum length of a Y-dominator sequence of \( G \) is called the Y-dominator sequence number of \( G \) and is denoted by \( l_Y(G) \).

Similarly one can define an X-dominator sequence number, \( l_X(G) \). The following theorem shows that the two numbers are equal.

**Proposition 3.2.** For any bipartite graph \( G \) we have \( l_X(G) = l_Y(G) \).

*Proof.* Let \( (x_1,x_2,\ldots,x_k) \) be a X-dominator sequence of \( G \) where \( k = l_X(G) \). For each \( i, 1 \leq i \leq k \), choose \( y_i \in N(x_i)-(N(x_1)\cup N(x_2)\cup \cdots \cup N(x_{i-1})) \). Clearly \( (y_k,y_{k-1},\ldots,y_1) \) is a Y-dominator sequence of \( G \). Hence \( l_X(G) \leq l_Y(G) \). By a similar argument \( l_X(G) \geq l_Y(G) \), and hence \( l_X(G) = l_Y(G) \). \( \square \)

In view of Proposition 3.2, we write \( l(G) = l_X(G) = l_Y(G) \) and \( l(G) \) is called the dominator sequence number of \( G \).

The study of dominator sequence had its origin in domination game as given in Brešar et al. [3]. Further the concept of dominating sequence and Grundy domination number of a graph was introduced in Brešar et al. [2]. In this definition the dominating sequence is not necessarily an independent set but forms a dominating set, whereas in our definition the dominator sequence is always independent and is not necessarily a dominating set. Thus the Grundy domination number \( \gamma_{gr}(G) \) and the dominator sequence number \( l(G) \) are different. For example for the complete bipartite graph \( G = K_{r,s} \) with \( r,s \geq 2, l(G) = 1 \) and \( \gamma_{gr}(G) = 2 \). Further motivated by the SLTM-problem described in Section 2, we confine ourselves to bipartite graphs. Basic results on independent dominator sequence number in arbitrary graphs are given in Jayaram et al. [1].

We observe that \( l(G) = 1 \) if and only if \( G \) is isomorphic to \( K_{r,s} \), with \( r,s \geq 1 \).

For the path \( P_n \) we have, \( l(P_n) = \lceil \frac{n}{2} \rceil \). For the bipartite complement \( P_{nc}^b \) of the path \( P_n = (v_1,v_2,\ldots,v_n) \), with \( n \geq 6 \), for any vertex \( v_i \) at most two vertices in the other partite set are not dominated by \( v_i \). Thus \( l(P_{nc}^b) = 3 \).
Proposition 3.3. For any even cycle \(C_n\) we have \(l(C_n) = \frac{n}{2} - 1\).

Proof. Let \(C_n = (x_1, x_2, \ldots, x_n, x_1)\). Obviously \((x_1, x_3, \ldots, x_{n-3})\) is a dominator sequence and hence \(l(C_n) \geq \frac{n}{2} - 1\). Further, any dominator sequence of length \(k\) in \(X\) dominates at least \(k + 1\) vertices in \(Y\) so that \(k + 1 \leq \frac{n}{2}\). Hence \(k \leq \frac{n}{2} - 1\). Thus \(l(C_n) = \frac{n}{2} - 1\). □

Proposition 3.4. For any even cycle \(C_n\), we have \(l(C_n \square K_2) = n - 2\).

Proof. Let \(C_n^0 = (x_1, x_2, \ldots, x_n, x_1)\) and \(C_n^1 = (y_1, y_2, \ldots, y_n, y_1)\) be the two copies of \(C_n\) in \(G\) with \(x_iy_j \in E(G)\). Then \(X = \{x_i : 1 \leq i \leq n\} \cup \{y_i : 1 \leq i \leq n\}\) and \(Y = V(G) - X\) is the bipartition of \(G\). Now \((x_1, y_2, x_3, y_4, \ldots, x_{n-3}, y_{n-2})\) is a dominator sequence of \(G\) and hence \(l(G) \geq n - 2\). Further any dominator sequence of length \(k\) in \(X\) dominates at least \(k + 2\) vertices in \(Y\), so that \(k + 2 \leq n\). Thus \(l(G) \leq n - 2\) and hence \(l(G) = n - 2\). □

Proposition 3.5. For any path \(P_n\), we have \(l(P_n \square K_2) = n - 1\).

Proof. The proof is similar to that of Proposition 3.4. □

Proposition 3.6. Let \(n\) and \(k\) be two positive integers with \(n > 2k\). Then there exists a bipartite graph \(G\) of order \(n\) with \(l(G) = k\).

Proof. Let \(A = \{x_1, x_2, \ldots, x_k\}\), \(B = \{y_1, y_2, \ldots, y_k\}\) and \(C = \{z_1, z_2, \ldots, z_{n-2k}\}\). Let \(G\) be the bipartite graph with bipartition \(X = A\) and \(Y = B \cup C\), where \(N(x_i) = \{y_i\} \cup C, 1 \leq i \leq k\). Clearly \((x_1, x_2, \ldots, x_k)\) is a dominator sequence of \(G\) and since \(|X| = k\), it follows that \(l(G) = k\). □

Lemma 3.7. For any bipartite graph \(G\) we have \(l(G) \leq \beta_1(G)\), where \(\beta_1(G)\) is the cardinality of a maximum matching in \(G\).

Proof. Let \((x_1, x_2, \ldots, x_k)\) be a dominator sequence of \(G\), where \(k = l(G)\). For each \(x_i\) choose \(y_i \in N(x_i) - (N(x_1) \cup N(x_2) \cup \cdots \cup N(x_{i-1}))\). Clearly, \(x_1y_1, x_2y_2, \ldots, x_ky_k\) is a matching in \(G\) and hence \(l(G) \leq \beta_1(G)\). □

Lemma 3.8. Let \(G\) be an \(r\)-regular bipartite graph of order \(n\). Then \(l(G) \leq \frac{n}{2} - r + 1\).

Proof. Let \((x_1, x_2, \ldots, x_k)\) be any dominator sequence of \(G\) with \(k = l(G)\). Since \(x_1\) dominates exactly \(r\) vertices in \(Y\) and each \(x_i\) dominates at least one vertex in \(Y\) not dominated by \(x_1, x_2, \ldots, x_{i-1}\), it follows that \(r + (k - 1) \leq |Y| = \frac{n}{2}\). Hence \(l(G) \leq \frac{n}{2} - r + 1\). □

Remark 3.9. The inequality given in the above theorem can be sharp and also strict. For example, consider the graph in Fig.2(a) with \(n = 14\) and \(r = 3\). Then \((1, 5, 6, 3, 7)\) is a dominator sequence of length, \(l(G) = \frac{n}{2} - r + 1 = 5\). For the graph in Fig.2(b) with \(n = 12\) and \(r = 3\), \((1, 2, 4)\) is a sequence of length, \(l(G) = 3 < \frac{n}{2} - r + 1\).

Proposition 3.10. Let \(G\) be any bipartite graph. Then for the shadow graph \(G_s\) of \(G\), we have \(l(G_s) = 2l(G)\).
Proof. Let $X, Y$ be a bipartition of $G$. Let $X'$ and $Y'$ be the sets of vertices in $G_s$ corresponding to $X$ and $Y$ respectively. Then $X \cup X', Y \cup Y'$ is a bipartition of $G_s$. Now if $l(G) = k$ and $(x_1, x_2, \ldots, x_k)$ is a dominator sequence of $G$, then $(x'_1, x'_2, \ldots, x'_k, x_1, x_2, \ldots, x_k)$ is a dominator sequence of $G_s$. Hence $l(G_s) \geq 2k = 2l(G)$. Now, let $S$ be any dominator sequence of $G_s$. Since the induced subgraphs $(X \cup Y')$ and $(X' \cup Y)$ are both isomorphic to $G$ and the vertices in $Y'$ are dominated only by the vertices in $X$, it follows that $|S \cap X'| \leq k$ and $|S \cap X| \leq k$. Hence $l(G_s) \leq 2k$. □

Proposition 3.11. Let $G$ be a connected graph of order $n$. Then for the subdivision graph $S(G)$ of $G$, we have $l(S(G)) = n - 1$.

Proof. The bipartition of $S(G)$ is given by $X = V(G)$ and $Y = V(S(G)) - V(G)$. We consider the dominator sequence to contain vertices from $Y$. Since any vertex of $Y$ has degree 2 in $S(G)$, any dominator sequence of length $k = \ell(S(G))$ dominates at least $k + 1$ vertices of $V(G)$ and hence $k + 1 \leq n$. Hence $l(S(G)) \leq n - 1$. Now, let $T$ be any spanning tree of $G$ with $E(T) = \{e_1, e_2, \ldots, e_{n-1}\}$. Let $y_i$ be the vertex in $Y$ that subdivides the edge $e_i$. Clearly $(y_1, y_2, \ldots, y_{n-1})$ is a dominator sequence of $S(G)$ and hence $l(G) \geq n - 1$. □

We now proceed to obtain a characterization of all bipartite graphs with $l(G) = |X|$. We first define recursively a family $\mathfrak{F}$ of bipartite graphs.

(i) All stars are in $\mathfrak{F}$.

(ii) If $G \in \mathfrak{F}$ and if $(X, Y)$ is a bipartition of $G$ with $|X| \leq |Y|$ and $S$ is any star and $G'$ is the graph obtained from $G$ and $S$ by joining the centre of $S$ to a nonempty subset of $Y$, then $G' \in \mathfrak{F}$.

Theorem 3.12. Let $G$ be a connected bipartite graph with bipartition $(X, Y)$ with $|X| \leq |Y|$. Then $l(G) = |X|$ if and only if $G \in \mathfrak{F}$.

Proof. Suppose $G \in \mathfrak{F}$. Let $X = \{x_1, x_2, \ldots, x_k\}$ be the set of all centres of the stars, in the order in which they appear, in the construction of $G$. Clearly $X, V(G) - X$ is a bipartition of $G$ and $X$ is a dominator sequence of $G$. Hence $l(G) = |X|$. Conversely, let $G$ be a bipartite graph with bipartition $X, Y$ such that $X = (x_1, x_2, \ldots, x_k)$ is a dominator sequence of $G$. Let $S_1 = N[x_1]$ and $S_i = N[x_i] - \bigcup_{j=1}^{i-1} N(x_j)$. Then $(S_i)$ is a star with centre $x_i$ and $G$ can be obtained from these stars by using the above construction. Hence $G \in \mathfrak{F}$. □
Definition 3.13. A bipartite graph $G$ is said to be dominator sequence critical if $l(G) = k$ and $l(G - v) < k$ for all $v \in V(G)$.

Proposition 3.14. A bipartite graph $G$ is dominator sequence critical if and only if $G$ has a perfect matching and $l(G) = |X|$.

Proof. Let $X, Y$ be a bipartition of $G$ and let $l(G) = k$. Let $(x_1, x_2, \ldots, x_k)$ be a dominator sequence of $G$. If $G$ has a perfect matching and $l(G) = |X|$, then $|X| = |Y| = k$ and trivially $l(G - v) < k$ for all $v \in V$. Conversely, suppose $l(G - v) < k$ for all $v \in V$. If $|X| > k$, then $l(G - x) = l(G)$ for any $x \in X - S$, where $S$ is a dominator sequence of $G$, a contradiction. Thus $|X| = k$. By a similar argument we have $|Y| = k$. Further if $\{x, y : y \in N(x) - \bigcup_{j=1}^{i-1} N(x_j) \text{ and } 1 \leq i \leq k\}$ is a perfect matching in $G$. \qed

Definition 3.15. Let $S = (x_1, x_2, \ldots, x_r)$ be a dominator sequence in a bipartite graph $G$. Let $y_i$ be a vertex which is dominated by $x_i$ but not dominated by $x_1, x_2, \ldots, x_{i-1}$. The matching $M = \{x_1y_1, x_2y_2, \ldots, x_ry_r\}$ is called a matching determined by $S$.

Definition 3.16. Let $G = (V, E)$ be a graph and $M$ a matching. An $M$-alternating path in $G$ is a path whose edges are alternatively in $E \setminus M$ and in $M$. An $M$-alternating path whose two end vertices are adjacent is called an $M$-alternating cycle.

The following result gives us a good characterization to identify a maximum dominator sequence based on the size of an $M$-alternating cycle in a graph. The following theorem is used later in the paper to identify dominator sequences.

Theorem 3.17. A maximum matching $M$ in a bipartite graph $G$ is determined by a dominator sequence $S$ in $G$ if and only if there does not exist an $M$-alternating cycle in $G$.

Proof. Suppose there exists a dominator sequence $S = \{x_1, x_2, \ldots, x_r\}$ which determines the matching $M = \{x_1y_1, x_2y_2, \ldots, x_ry_r\}$. Let $H$ be the subgraph of $G$ induced by $V(M) = \{x_1, x_2, \ldots, x_r, y_1, y_2, \ldots, y_r\}$. Since $x_1$ dominates $y_1$, but does not dominate $y_2, y_3, \ldots, y_r$, it follows that $x_1$ has degree 1 in $H$. Hence the edge $x_1y_1$ does not lie on an $M$-alternating cycle in $G$. Now $x_2$ is a vertex of degree 1 in $H - \{x_1, y_1\}$ and hence the edge $x_2y_2$ does not lie in an $M$-alternating cycle. By repeating this process, we see that no edge of $M$ lies on an $M$-alternating cycle and hence there does not exist an $M$-alternating cycle in $G$.

Conversely, suppose that $M = \{x_1y_1, x_2y_2, \ldots, x_ry_r\}$ is a maximum matching in $G$ such that there does not exist an $M$-alternating cycle in $G$. Let $H$ be the subgraph of $G$ induced by $V(M) = \{x_1, x_2, \ldots, x_r, y_1, y_2, \ldots, y_r\}$ and let $P = (u = v_1, v_2, \ldots, v_s = v)$ be an $M$-alternating path of maximum length in $H$. Then the length of $P$ is odd, the first edge $v_1v_2$ and the last edge $v_{s-1}v_s$ of $P$ are in $M$ and the vertices $v_1, v_s$ are not adjacent to any vertex in $V(H) - V(P)$. Further since there does not exist an $M$-alternating cycle in $G$, the vertices $v_1, v_s$ are not adjacent to any vertex in $V(P)$. Thus both $v_1$ and $v_s$ are pendant vertices in $H$ and since the length of $P$ is odd, one of these vertices, say $v_1$, is in $X$. Let $v_1 = x_1, D_1 = \{x_1\}, M_1 = \{x_1y_1\}$ and $H_1 = H - \{x_1, y_1\}$. Now by considering an $M$-alternating path of maximum length in $H_1$, we obtain a vertex, say $x_2$, of degree 1 in $X \cup V(H_1)$. Let $D_2 = \{x_1, x_2\}, M_2 = \{x_1y_1, x_2y_2\}$ and $H_2 = H_1 - \{x_2, y_2\}$. By repeating
this process we obtain a dominator sequence \( D = \{x_1, x_2, \ldots, x_r\} \) in \( G \) which determines the matching \( M \).

In the following theorem we give bounds on the number of edges of a bipartite graph \( G \) of order \( n \) with \( l(G) = k \).

**Theorem 3.18.** Let \( G \) be a bipartite graph of order \( n \) and size \( m \) with bipartition \( X, Y \), where \( r = |X| \leq |Y| = s \) and let \( l(G) = k \). Then \( n - 1 \leq m \leq rs - \binom{k}{2} \). Further, \( m = n - 1 \) if and only if \( G \) is a tree with \( \beta_1(G) = k \). Also, \( m = rs - \binom{k}{2} \) if and only if \( G \) is isomorphic to the graph \( G_1 \) obtained from the complete bipartite graph \( K_{r,s} \) with bipartition \( X = \{x_1, x_2, \ldots, x_r\} \) and \( Y = \{y_1, y_2, \ldots, y_s\} \) by deleting all edges \( x_iy_j \) where \( 1 \leq i < j \leq k \).

**Proof.** Let \( (x_1, x_2, \ldots, x_k) \) be a dominator sequence in \( G \) and let \( y_i \in N(x_i) - \bigcup_{j=1}^{i-1} N(x_j) \), \( 1 \leq i \leq k \). Then \( x_iy_j \notin E(G) \) for all pairs \( i, j \) with \( 1 \leq i < j \leq k \). Hence \( m \leq rs - \binom{k}{2} \). Since \( G \) is connected, the lower bound is trivial. Now suppose \( l(G) = k \) and \( m = n - 1 \). Then it follows from Theorem 5.1 that \( G \) is a tree and \( \beta_1(G) = k \). Conversely, if \( G \) is a tree with \( \beta_1(G) = k \) then trivially \( m = n - 1 \).

Now let \( l(G) = k \) and \( m = rs - \binom{k}{2} \). Let \( (x_1, x_2, \ldots, x_k) \) be a maximum dominator sequence and let \( y_i \in N(x_i) - \bigcup_{j=1}^{i-1} N(x_j) \). Clearly \( x_i \) is not adjacent to \( y_{i+1}, y_{i+2}, \ldots, y_k \), \( 1 \leq i < k \) and since \( m = rs - \binom{k}{2} \) it follows that \( G = G_1 \). Conversely, suppose \( G \) is isomorphic to the graph \( G_1 \) given in the theorem. Clearly, \( m = rs - \binom{k}{2} \). Also, \( (x_1, x_2, \ldots, x_k) \) is a dominator sequence of \( G_1 \) and hence \( l(G) \geq k \). Further since \( N(x_i) = Y \) for all \( i \geq k \), it follows that \( l(G_1) = k \). \( \square \)

### 4 Complexity Results

In this section we prove that the decision problem corresponding to the dominator sequence number is NP-Complete.

**DOMINATOR SEQUENCE PROBLEM (DSP)**

**INSTANCE:** A bipartite graph \( G \) with bipartition \( X, Y \) and a positive integer \( k \).

**QUESTION:** Is there a sequence \( x_{i_1}, x_{i_2}, \ldots, x_{i_k} \) of vertices in \( X \) such that \( N(x_{i_1}) \setminus (N(x_{i_1}) \cup N(x_{i_2}) \cup \cdots \cup N(x_{i_{j-1}})) \neq \emptyset \) for all \( 2 \leq j \leq k \)?

For this purpose we use the feedback vertex set problem.

**FEEDBACK VERTEX SET**

**INSTANCE:** Graph \( G = (V, A) \) and a positive integer \( k \leq |V| \).

**QUESTION:** Is there a subset \( V' \subseteq V \) with \( |V'| \leq k \) such that \( V' \) contains at least one vertex from every cycle in \( G \)?

Karp [8] has proved that FEEDBACK VERTEX SET is NP-complete. We now define another problem called closure problem.

**Definition 4.1.** Let \( X \) be a set and let \( C \) be a collection of subsets of \( X \). For a subset \( A \subseteq X \) the closure of \( A \), denoted by \( \sigma(A) \) is the minimal subset \( B \) of \( X \) satisfying the following properties:

\[ \begin{align*}
\sigma(A) & \supseteq A \\
\sigma(A) & \text{is closed under inclusion} \\
\sigma(A) & \text{is closed under union} \\
\sigma(A) & \text{is closed under intersection} \\
\sigma(A) & \text{is the smallest such set}
\end{align*} \]
1 $A \subseteq B$.

2 There is no set $P \in C$ such that $|P \setminus B| = 1$.

**Observation 4.2.** Let $\mathcal{F} = \{B \subseteq X : B \supseteq A \text{ and there does not exist } P \in C \text{ such that } |P \setminus B| = 1\}$. Clearly $X \in \mathcal{F}$ and if $B_1, B_2 \in \mathcal{F}$, then $B_1 \cap B_2 \in \mathcal{F}$. Hence the closure of $A$ is given by $\sigma(A) = \bigcap_{B \in \mathcal{F}} B$, so that $\sigma(A)$ is well defined.

**CLOSED PROBLEM (CP)**

**INSTANCE:** A set $X$, a collection $C$ of subsets of $X$ and a positive integer $k$.

**QUESTION:** Is there a subset $A \subseteq X$ with $|A| \leq k$ and $\sigma(A) = X$.

We first prove that the CLOSURE PROBLEM is NP-complete by reduction from feedback vertex set. We finally prove that dominator sequence problem is NP-complete by reduction from closure problem.

**Theorem 4.3.** **CLOSED PROBLEM** is NP-Complete.

**Proof.** The proof is by reduction from FEEDBACK VERTEX SET PROBLEM. Let $(G, k)$ be an instance of the feedback vertex set problem. We may assume that the minimum degree of $G$ is at least two. With each vertex $v$ of $G$ we associate an element $x_v$ and with each neighbor $u$ of $v$ we associate two elements $x^1_{vu}, x^2_{vu}$. Note that the element $x^1_{vu}$ is different from the element $x^2_{vu}$.

Let $X = \{x_v : v \in V\} \cup \{x^1_{vu} : v \in V, u \in N(v)\} \cup \{x^2_{vu} : v \in V, u \in N(v)\}$. Clearly $|X| = n + 4m$ where $n = |V(G)|$ and $m = |E(G)|$. The sets in $C$ are constructed as follows. For every neighbor $u$ of $v$, there are three sets in $C$, $\{x_w : w \in N(v) \setminus \{u\}\} \cup \{x^1_{vu}\}, \{x_w : w \in N(v) \setminus \{u\}\} \cup \{x^2_{vu}\}$, and $\{x_v, x^1_{vu}, x^2_{vu}\}$. Thus the total number of sets is $6m$. Let $X_V = \{x_v : v \in V\}$.

We claim that $G$ has a feedback vertex set of size at most $k$ if and only if there exists a subset $A \subseteq X$ with $|A| \leq k$ and $\sigma(A) = X$. First suppose $F$ is a feedback vertex set of size at most $k$. Let $A = \{x_v : v \in F\}$. We claim that $\sigma(A) = X$. We first claim that $X_V \subseteq \sigma(A)$. Suppose $V' = \{v : x_v \notin \sigma(A)\}$ is not empty. Since $V'$ induces a forest in $G$, there exists a vertex $v \in V'$ which has at most one neighbor in $V'$. Let $u$ be a neighbor of $v$ such that all the neighbors of $v$ are not in $V'$. Then $\{x_w : w \in N(v) \setminus \{u\}\} \subseteq \sigma(A)$. Then this implies $x^1_{vu}, x^2_{vu} \in \sigma(A)$ and the set $\{x_v, x^1_{vu}, x^2_{vu}\} \subseteq A$, which implies that $x_v \in \sigma(A)$, a contradiction. If $X_V \subseteq \sigma(A)$, then it follows that all the elements $x^1_{vu}, x^2_{vu}$ must also be in $\sigma(A)$, and hence $\sigma(A) = X$.

Now let $A \subseteq X$ be a subset such that $\sigma(A) = X$ and $|A| \leq k$. Choose such a set $A$ such that $|A \setminus X_V|$ is minimum. Suppose $x^1_{vu} \in A$ for some vertex $v$ and neighbor $u$ of $v$. By the choice of $A$, $A' = \sigma(A \setminus \{x^1_{vu}\}) \subseteq X$. Note that $A'$ cannot contain $X_V$ otherwise $A' = X$. Since $\sigma(A' \cup \{x^1_{vu}\}) = X$, there must be a set $B$ such that $|B \setminus (A' \cup \{x^1_{vu}\})| = 1$. Thus the set $B$ must contain $x^1_{vu}$, some element $x \notin A' \cup \{x^1_{vu}\}$, and all other elements in $B$ are in $A'$. If $x = x_w$ for some $w \in V(G)$, let $A_1 = (A \cup \{x_w\}) \setminus \{x^1_{vu}\}$. Then $\sigma(A_1) = X$ and $A_1$ contains fewer elements not in $X_V$, contradicting the choice of $A$. Therefore $x = x^i_{pq}$ for some vertex $p$ and neighbor $q$ of $p$, and $i \in \{1, 2\}$. But there is a set that contains $x^i_{vu}$ and $x^i_{pq}$ if $v = p, u = q$ and $i = 2$. The only such set is $\{x_v, x^1_{vu}, x^2_{vu}\}$. This implies that $x_v \in A'$. Again since $\sigma(A \cup \{x^1_{vu}\} \cup \{x^2_{vu}\}) = X$, there is a set $B_1$ such
that $|B_1 \setminus (A' \cup \{x_{vu}^1 \} \cup \{x_{vu}^2 \})| = 1$, $B_1$ contains exactly one of $x_{vu}^1$ or $x_{vu}^2$ and other elements in $B_1$ are in $A'$. Then the element in $B_1 \setminus (A' \cup \{x_{vu}^1 \} \cup \{x_{vu}^2 \})$ must be $x_w$ for some vertex $w$. Now replace $x_{vu}^1$ in $A$ by $x_w$, to get a set containing fewer elements not in $X_V$, but whose closure is $X$, contradicting the choice of $A$.

So, we may assume $A \subseteq X_V$. We claim that $F = \{v|x_v \in A\}$ is a feedback vertex set in $G$. Let $V' = V(G) \setminus F$ and suppose there is a cycle $C$ in the subgraph induced by $V'$. We show that the closure of $A$ is a proper subset of $X$, a contradiction.

Consider the set $B' = \{x_v|v \in V(C)\} \cup \{x_{vu}, x_{vu}^2|(N(v) \setminus \{u\}) \cap V(C) \neq \emptyset, uv \in E(G)\}$ and let $B = X \setminus B'$. Then $B$ contains $A$ and is a proper subset of $X$. We show that $\sigma(B) = B$, giving a contradiction. To do that, it is sufficient to show that any set that contains an element of $B'$ contains at least two such elements.

Suppose some set contains an element $x_v \in B'$. The sets containing $x_v$ are of the form $\{x_v, x_{vu}^1, x_{vu}^2\}$ for some neighbor $u$ of $v$, or $\{x_v|x_p \in N(u) \setminus \{u\}\} \cup \{x_{vu}^i\}$ for some neighbor $u$ of $v$, a neighbor $u$ of $v \neq v$, and $i \in \{1,2\}$. In the first case, both $x_{vu}^1, x_{vu}^2 \in B'$ as $v$ has a neighbor other than $u$ in the cycle. Similarly, $x_{vu} \in B'$ since $u$ has a neighbor other than $w$ in the cycle. Suppose $B'$ contains an element $x_{vu}^i$. Then $v$ has a neighbor $w$ other than $u$ in $C$. The sets containing $x_{vu}^i$ are $\{x_v, x_{vu}^1, x_{vu}^2\}$ and $\{x_v|x_p \in N(u) \setminus \{u\}\} \cup \{x_{vu}^i\}$. In the first case, both $x_{vu}^1, x_{vu}^2$ must be in $B'$ and in the second case $x_{vu}$ is in $B'$ since $w \in V(C)$ and $w \neq u$.

Therefore $B$ satisfies the required properties and $\sigma(B) = B$. This contradicts $\sigma(A) = X$. Thus $F$ must be a feedback vertex set of size at most $k$.

**Theorem 4.4. Dominator Sequence Problem (DSP) is NP-Complete.**

**Proof.** The reduction is from CLOSURE PROBLEM. Consider an instance of the closure problem $X,C = \{C_1, C_2, \ldots, C_l\} \cup \{X\}$ and $|X| - k$. Let $G$ be the bipartite graph with bipartion $X,Y$, where $|Y| = l$ and $\{N(y_i) : y_i \in Y\} = C_i, 1 \leq i \leq l$. Let $A' = (x_{i_1}, x_{i_2}, \ldots, x_{i_k})$ be a sequence of vertices in $X$, and $A = X \setminus A'$ such that the closure $\sigma(A) = X$. We claim that $N(x_{i_j}) \setminus (N(x_{i_1}) \cup N(x_{i_2}) \cup \ldots \cup N(x_{i_{j-1}})) \neq \emptyset$ for all $2 \leq j \leq k$. We note that $|C_i \setminus X| = 0, 1 \leq i \leq l$, since the closure of $A$ is the entire set $X$. Suppose on the contrary, let $x_{i_j} \in A'$ be a vertex that does not satisfy the property. Then, deleting $x_{i_j}$ from $X$, we note that for any $C_i$, if $x_{i_j} \notin C_i$ then $|C_i \setminus X| = 0$, contradicting the minimality of $X$ and if $x_{i_j} \in C_i$, then removing $x_{i_j}$ from $X$, also removes $x_{i_j}$ from $C_i$, and hence $|C_i \setminus X| = 0$, again contradicting the minimality of $X$. Hence $\{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\}$ is a dominator sequence of $G$. Conversely, suppose $G$ has a dominator sequence $(x_{i_1}, x_{i_2}, \ldots, x_{i_k})$. Then $N(x_{i_j}) \setminus (N(x_{i_1}) \cup N(x_{i_2}) \cup \ldots \cup N(x_{i_{j-1}})) \neq \emptyset$ for all $2 \leq j \leq k$. Let $A' = \{x_{i_j}|1 \leq j \leq k\}$ and let $A = X \setminus A'$. Then we claim that the closure of $A$ is $X$. Suppose not and let $j$ be the largest number $\leq k$ such that $x_{i_j} \notin \sigma(A)$. If $j = 1$, then $|X \setminus \sigma(A)| = 1$ and since $X \subseteq C$, this contradicts the definition of $\sigma(A)$. Suppose $j \geq 2$ and let $y_{i_j}$ be an element in $N(x_{i_j}) \setminus N(x_{i_1}) \cup N(x_{i_2}) \cup \ldots \cup N(x_{i_{j-1}})$. Then $N(y_{i_j})$ is a set in $C$ such that $|N(y_{i_j}) \setminus \sigma(A)| = 1$, a contradiction. Thus $\sigma(A) = X$. \qed

## 5 Dominator Sequences in Trees

In this section we present a linear time algorithm to obtain a maximum dominator sequence for trees. It follows from Theorem 3.17 that for any tree $T$, we have $l(T) = \beta_1(T)$.
Theorem 5.1. For any tree $T$, a dominator sequence $D$ of $T$ with $|D| = l(T) = \beta_1(T)$ can be determined in linear time.

**Proof.** Let $X, Y$ be the bipartition of $T$. A maximum matching $M$ of $T$ with $|M| = \beta_1(T)$ can be obtained in linear time (See problem 7.95 in [10]). Let $T$ be a tree and let $M$ be a maximum matching in $T$. Let $T'$ be a BFS tree of $T$ rooted at a vertex $r$. For each vertex $v \in V(T)$, let $h(v)$ denote the level of $v$ in $T'$. Let $V(M)$ be the vertex induced subgraph of $M$ in $T'$. Clearly $V(M)$ is a forest and $\forall v \in V(M), v$ is a matched vertex. Let $T_0 = V(M), D = \emptyset$ and $h_l$ be the highest level such that the vertices at level $h_l$ belong to the $X$ partition of $T'$. For each pendant vertex $v$ at level $h_l$, let $D = D \cup \{v\}, T' = T' - \{uv\}$, where $u$ is a matched vertex with $v$ at level $h_l - 1$. Since the set of all vertices at level $h_l$ are independent and $V(M)$ is a vertex induced matching, each vertex $v$ at level $h_l$ dominates a new vertex at level $h_l - 1$. Again, for each pendant vertex $v$ at level $h_l - 2$, let $D = D \cup \{v\}, T' = T' - \{uv\}$. We repeat this process for all alternate levels until either level 1 or level 2 is reached and repeat the same process starting from level 1 or level 2 until level $h_l$ is reached or until there are no more matching edges remaining. Clearly, $D$ is a dominator sequence and since we process all edges of the tree at most once, it is easy to see that $D$ can be obtained in $O(n)$ time. \hfill \Box

The following algorithm gives a maximum dominator sequence for any given tree, $T$ in linear time.

**Algorithm : Tree Dominator Sequence (TDS)**

**Input:** A tree $T(X, Y)$ with $|V(T)| = n$.

**Output:** A maximum dominator sequence $L(T) = (x_1, x_2, \ldots, x_k)$.

1. Initialize $L(T) = \emptyset$, $X'$ to be an empty stack. Select a vertex $r$ as root vertex. Assign levels to all the vertices in $T$ with the level of $r$ as zero. Let $l$ be the highest level.

2. Compute a maximum matching $M = \{x_1y_1, x_2y_2, \ldots, x_ky_k\}$ of $T$, where $k = \beta_1(T)$.

3. For level $j = l, l - 1, l - 2, \ldots, 0$ do

   3.1. For each $x_i$ at level $j$

      3.1.1. If $\deg(x_i) = 1$, then append $x_i$ to $L(T)$. Assign $\text{UNIQUE}(x_i) = y_i$ where $y_i \in N(x_i)$ and remove $x_i$ and $y_i$.

      3.1.2. If there exists a pendant vertex $v \in N(x_i)$, then add $x_i$ to $X'$, assign $\text{UNIQUE}(x_i) = v$ and remove $x_i$ and $v$.

4. For each element $x_i$ on top of $X'$ append $x_i$ to $L(T)$.

5. Output $L(T)$.

**Complexity:** Assigning levels to each vertex can be done through a breadth first search of the tree. An $O(n)$ linear time algorithm for computing a maximum matching in trees is known (See problem 7.95 in [10]). In step 3, each vertex of the partite set $X$ in the matching is processed and hence the complexity is the sum of the degrees, which in the case of trees is $O(n)$. Hence, the complexity of finding a maximum dominator sequence in trees is $O(n)$. We need the following properties that arise from the breadth first search $T$ used to assign levels to the vertices.
P1: A vertex \( v \neq r \) at level \( j \) is adjacent to exactly one vertex at level \( j - 1 \).

P2: No vertex \( v \) at level \( j \) is adjacent to any vertex at any level higher than \( j + 1 \).

P3: All vertices on a given level form an independent set.

In the following, we shall refer to vertices identified in step 3.1.1 as \textit{type-1} vertices and vertices identified in step 3.1.2 as \textit{type-2} vertices.

\textbf{Theorem 5.2.} \textit{The sequence} \( L(T) \) \textit{given by the algorithm TDS is a maximum dominator sequence of} \( T \).

\textit{Proof.} We prove by showing that \( \text{UNIQUE}(x_i) \) is the vertex adjacent to \( x_i \) and not adjacent to any \( x_j \) that occurs earlier to \( x_i \) in the sequence \( L(T) \). It follows from P1 that if \( x_i \) and \( x_j \) are of \textit{type-1} and are at the same level, then \( \text{UNIQUE}(x_i) \) is not adjacent to \( x_j \) and \( \text{UNIQUE}(x_j) \) is not adjacent to \( x_i \). Now, suppose \( x_i \) and \( x_j \) that are of \textit{type-2}. If \( x_i \) and \( x_j \) are at the same level then, \( \text{UNIQUE}(x_i) \) is not adjacent to \( x_j \) and \( \text{UNIQUE}(x_j) \) is not adjacent to \( x_i \). If \( x_j \) and \( x_i \) are at different levels and the level of \( x_j \) is lower than the level of \( x_i \), then step 4 of the algorithm TDS ensures that \( x_j \) occurs before \( x_i \) in \( L(T) \). In this case \( x_j \) is at least two levels lower than \( \text{UNIQUE}(x_i) \), and hence by P2, \( x_j \) is not adjacent to \( \text{UNIQUE}(x_i) \).

If \( x_i \) is of \textit{type-1} and \( x_j \) is of \textit{type-2}, then \( x_i \) occurs before \( x_j \) in \( L(T) \) and it follows from P1, P2 and P3 that \( \text{UNIQUE}(x_j) \) is not adjacent to \( x_i \).

Since all \( x_i \)'s are in \( L(T) \) and \( |L(T)| = |M| = \beta_1(T) \), \( L(T) \) is a maximum dominator sequence. \( \square \)

\section{Conclusion}

Motivated by an application in the design of survivable logical topology mapping (SLTM) in an IP-over-WDM optical network, we defined the concept of dominator sequence in a bipartite graph and initiated a study of the corresponding parameter.

The concept of dominator sequence can be extended to general graphs. For instance, we can define a set \( X = \{x_1, \ldots, x_k\} \) to be a dominator sequence of a graph \( G = (V, E) \) if \( X \) is an independent set and each \( x_i \in X \) is adjacent to at least one vertex not dominated by \( x_1, x_2, \ldots, x_{i-1} \).

Another definition of dominator sequence for general graphs is to drop the requirement that the set \( X \) be independent. In fact, such a sequence called, \textit{incident cover sequence}, was defined in [13] and [2] and was used in [15] to study the problem of augmenting the logical graph with additional edges to guarantee that the augmented graph admits a survivable mapping.

We believe that further study of the dominator sequence concept for general and bipartite graphs will have significant theoretical as well as practical implications for the survivable logical topology mapping problem.

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