Dominator Sequences in Bipartite Graphs

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Abstract

Let $G = (V, E)$ be a bipartite graph with bipartition X, Y and let $|X| \leq |Y|$. A dominator sequence in G is a sequence of vertices $(x_1, x_2, \ldots, x_k)$ in X such that for each i with $2 \leq i \leq k$, the vertex x_i dominates at least one vertex in Y which is not dominated by $x_1, x_2, \ldots, x_{i-1}$. The maximum length of a dominator sequence in G is called the dominator sequence number of G and is denoted by $l(G)$. In this paper we present several basic results on this parameter. We prove that the decision problem for the parameter $l(G)$ is NP-Complete. We obtain bounds for $l(G)$ and discuss applications in the study of optical networks.

Keywords : Bipartite graph, domination number, dominator sequence number, optical networks, survivability of IP-over-WDM networks.

1 Introduction

Throughout this paper $G = (V, E)$ stands for a bipartite graph with bipartition X, Y and we assume that G does not have isolated vertices. For graph theoretic terminology we refer to Chartrand and Lesniak [4].

Hedetniemi and Laskar [6, 7] proposed a bipartite theory of graphs and suggested an equivalent formulation of several concepts on graphs as concepts for bipartite graphs. One among them is the concept of Y-domination, where for a connected bipartite graph

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$G = (V, E)$ with bipartition X, Y, a subset D of X is a *Y-dominating set* of G if every $y \in Y$ is adjacent to at least one vertex of D. The minimum order $\gamma_Y(G)$ of a Y-dominating set of G is called the *Y-domination number* of G. In this paper we introduce the concept of dominator sequence number of a bipartite graph, which is motivated by an application in optical networks. We present several basic results on this concept. We establish the NP-Completeness of the problem of finding the *dominator sequence number* of bipartite graphs and we give a linear time algorithm for constructing a maximum dominator sequence in a tree. Other possible generalizations of the concept of dominator sequence are given in Section 6. We need the following definitions.

Definition 1.1. *A Cartesian product $G = G_1 \Box G_2$ has $V(G) = V(G_1) \times V(G_2)$, and two vertices (u_1, u_2) and (v_1, v_2) of G are adjacent if and only if either $u_1 = v_1$ and $u_2v_2 \in E(G_2)$ or $u_2 = v_2$ and $u_1v_1 \in E(G_1)$.*

Definition 1.2. *Let G be a connected graph. The subdivision graph $S(G)$ of a graph G is the graph obtained from G by replacing each edge uv of G by a new vertex w along with edges uw and vw.*

Definition 1.3. *For a graph $G = (V, E)$, the shadow graph of G is the graph G_s with vertex set $V \cup V'$ where $V' = \{x' : x \in V\}$ and is disjoint from V, and edge set $E' = E \cup \{xy' : xy \in E\}$.*

Definition 1.4. *Let G be a bipartite graph with bipartition X, Y. The bipartite complement G^{bc} of G has the same bipartition X, Y and $x_iy_j \in E(G^{bc})$ if and only if $x_iy_j \notin E(G)$.*

2 Motivation: Survivable Logical Topology Mapping in an IP-over-WDM Optical Network

The concept of layering plays an important role in the design of communication networks and protocols. An IP(Internet Protocol) over WDM(Wavelength Division Multiplexing) network is an example of a layered network. Here the WDM optical network is the physical layer represented by a graph G_p. The IP layer is the logical layer represented by a graph G_l. Without loss of generality we assume that G_l has the same vertex set as G_p. Also we assume that G_l and G_p are both 2-edge connected.

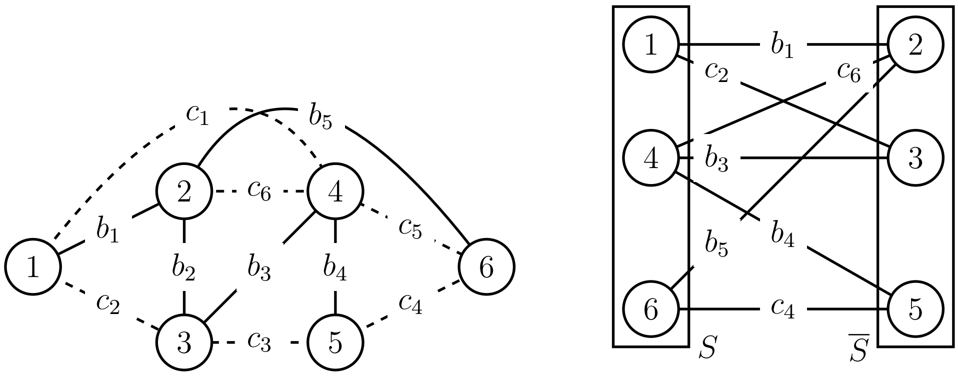
Each edge in G_l between vertices v nd w corresponds to a path(called lightpath) between v and w in G_p[5]. To transmit information from vertex u to vertex v, first a u-v path P in G_l is identified. Then the information is transmitted using lightpaths corresponding to the logical links in P. If an edge in G_p fails, then several edges in G_l could fail causing G_l to become disconnected and thereby disrupting transmission of information. Survivable logical topology mapping (SLTM) is to map each edge in G_l into a lightpath in G_p such that a single edge failure in G_p does not disconnect G_l.

The SLTM problem has been studied using two approaches. The approach using mathematical programming formulation was pioneered in [12]. The other approach, called the structural approach, uses graph-theoretic concepts and was pioneered in [9].

The structural approach (though described differently in [9]) can be explained using the concept of ear decomposition of G_l. This approach may be viewed as constructing an ear decomposition of G_l and mapping the edges in each ear into edge-disjoint lightpaths in G_p. If no ear decomposition that admits such a mapping is available then the given SLTM problem is infeasible. For details of this approach see [9], [13] and [14].

For purposes of the application under discussion, our definition of a ear given next is slightly more general than the traditional definition of an ear given in [17]. The first ear is a circuit in the given graph. Each subsequent ear is a circuit obtained by contracting the other ears already selected. If an ear has exactly one edge , the corresponding lightpath in the physical graph G_p can be selected arbitrarily and so such ears are not of interest in the cross layer survivability mapping problem. Since circuits/cutsets, and edge contraction/deletion are dual concepts[11], the question arises whether there exists a dual cutset-based approach for the SLTM problem. This question has been discussed in [13] and [14]. A brief outline of the cutset-based approach is as follows.

Consider a connected, undirected and simple graph $G = (V, E)$ with vertex set V and edge set E. Let $|V| = n$ and $|E| = m$. Consider a partition $(S, \overline{S})$ of V, where $\overline{S} = V - S$. Then the set of edges with one end in S and the other in $\overline{S}$ is called a *cut* of G.



(a) (b)

Figure 1: (a)A graph with a spanning tree (bold lines) (b) A cut.

For example consider the graph G in Fig.1(a). Here the vertices are numbered $1, 2, \ldots, 6$. The bold edges in this figure denote the branches of a spanning tree T of G and the dotted edges are the chords of this tree. The partition $(S, \overline{S})$ with $S = \{1, 4, 6\}$ and $\overline{S} = \{2, 3, 5\}$ defines the cut shown in Fig.1(b). If we remove a branch b from a spanning tree T, then the tree T gets disconnected resulting in two trees (not spanning) T_1 and T_2. The sets of nodes in T_1 and T_2 define a partition of V. The corresponding cut is called the *fundamental cutset* of T with respect to the branch b. For example, if we remove the branch b_3 from the tree T of Fig.1(a) then we get trees T_1 and T_2 given by the branches $\{b_1, b_2, b_5\}$ and $\{b_4\}$, respectively. The corresponding fundamental cutset $Q(b_3)$ consists of the edges $\{b_3, c_1, c_3, c_4, c_5, c_6\}$. Note that the subgraphs induced by the vertex sets of T_1 and T_2 are both connected.

The *fundamental cutset matrix* with respect to the tree T is defined as $Q^f = [q_{ij}]_{(n-1)\times(m)}$. The matrix Q^f has $(n-1)$ rows, one for each fundamental cutset and one column for each edge. The entry q_{ij} is defined as

$$q_{ij} = \begin{cases} 1, & \text{if } Q(b_i) \text{ contains edge } j \\ 0, & \text{otherwise.} \end{cases}$$

Arranging the rows of Q^f such that the j^{th} row corresponds to f-cutset $Q(b_j)$ and the columns correspond to the edges in the order $\{b_1, b_2, \ldots, b_{n-1}, c_1, c_2, \ldots, c_{m-n+1}\}$ the matrix Q^f can be written as $Q^f = [U|Q^{fc}]$. For example, the Q^f matrix with respect to the tree T of Fig.1(a) is given below.

$$\begin{array}{c} \\ b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{array} \begin{array}{c} \begin{array}{ccccc|cccccc} b_1 & b_2 & b_3 & b_4 & b_5 & c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \end{array} \\ \left[\begin{array}{ccccc|cccccc} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right] \end{array} \longrightarrow (1)$$

An ordered sequence $Q(b_1), Q(b_2), \ldots, Q(b_k)$ is a *cutset cover sequence or simply a Q-sequence* of length k if

a) $[Q(b_j) - b_j - \bigcup_{p=1}^{j-1} Q(b_p)] \neq \emptyset, 2 \leq j \leq k$.

b) $\bigcup_{p=1}^{k} Q(b_p) = E-$ {branches not in the Q-sequence}.

Note that for a given spanning tree and its f-cutsets, there may be more than one Q-sequence. For example, for the fundamental cutsets given in (1), following are the three Q-sequences.

1. $Q(b_4), Q(b_5), Q(b_3), Q(b_2)$.
2. $Q(b_4), Q(b_5), Q(b_1), Q(b_2)$.
3. $Q(b_1), Q(b_2), Q(b_4)$.

Without loss of generality assume that $Q(b_1), Q(b_2), \ldots, Q(b_k)$ is a Q-sequence of length k. Let us define $\widehat{S}(b_j)$ as follows:

a) $\widehat{S}(b_1) = Q(b_1) - b_1$.

b) $\widehat{S}(b_j) = Q(b_j) - b_j - \bigcup_{p=1}^{j-1} Q(b_p), 2 \leq j \leq k$.

The sets $b_i \cup \widehat{S}(b_i)$ are called cutset-ears. The ears corresponding to the branches in a cutset cover sequence D form a cutset-ear decomposition of the graph obtained from G_l by contracting the branches that are not in the cutset cover sequence.

The cutset-based([13],[14]) method has the following steps:

1. Map the edges in each cutset-ear $b_i \cup \widehat{S}(b_i)$ into mutually edge-disjoint paths in G_p.
2. For each branch b_i not in the cutset-cover sequence, add a parallel edge in G_l and map these edges into mutually edge-disjoint paths in G_p. The newly added edges are called protection edges.

Each protection edge entails the provision of additional resources in G_p and so is expensive. So, to minimize this cost, we need to start with a cutset-cover sequence which has the largest number of branches possible.

We now construct a bipartite graph $G(X, Y)$ where each vertex in X corresponds to a branch b_i of a spanning tree T, each vertex in Y corresponds to a chord c_i and b_i is joined to c_j if and only if the fundamental cutset with respect to b_i contains the chord c_j. Then a cutset-cover sequence in G corresponds to a dominator sequence, which we formally define in the next section. The SLTM problem requires the construction of a longest dominator sequence.

3 Basic Results

In this section we introduce the concept of a dominator sequence in a bipartite graph and present several basic results.

Definition 3.1. *Let G be a bipartite graph with bipartition X, Y. A Y-dominator sequence is a sequence of vertices $(x_1, x_2, \ldots, x_k)$ in X such that for each $i, 1 \leq i \leq k$, there exists $y_i \in Y$ such that y_i is dominated by x_i and y_i is not dominated by any of the vertices $x_1, x_2, \ldots, x_{i-1}$, or equivalently, $y_i \in N(x_i) - (N(x_1) \cup N(x_2) \cup \cdots \cup N(x_{i-1}))$. The maximum length of a Y-dominator sequence of G is called the Y-dominator sequence number of G and is denoted by $l_Y(G)$.*

Similarly one can define an X-dominator sequence number, $l_X(G)$. The following theorem shows that the two numbers are equal.

Proposition 3.2. *For any bipartite graph G we have $l_X(G) = l_Y(G)$.*

Proof. Let $(x_1, x_2, \ldots, x_k)$ be a X-dominator sequence of G where $k = l_X(G)$. For each $i, 1 \leq i \leq k$, choose $y_i \in N(x_i) - (N(x_1) \cup N(x_2) \cup \cdots \cup N(x_{i-1}))$. Clearly $(y_k, y_{k-1}, \ldots, y_1)$ is a Y-dominator sequence of G. Hence $l_X(G) \leq l_Y(G)$. By a similar argument $l_X(G) \geq l_Y(G)$, and hence $l_X(G) = l_Y(G)$. □

In view of Proposition 3.2, we write $l(G) = l_X(G) = l_Y(G)$ and $l(G)$ is called the *dominator sequence number of G*.

The study of dominator sequence had its origin in domination game as given in Brešar et al. [3]. Further the concept of dominating sequence and Grundy domination number of a graph was introduced in Brešar et al. [2]. In this definition the dominating sequence is not necessarily an independent set but forms a dominating set, whereas in our definition the dominator sequence is always independent and is not necessarily a dominating set. Thus the Grundy domination number $\gamma_{gr}(G)$ and the dominator sequence number $l(G)$ are different. For example for the complete bipartite graph $G = K_{r,s}$ with $r, s \geq 2, l(G) = 1$ and $\gamma_{gr}(G) = 2$. Further motivated by the SLTM-problem described in Section 2, we confine ourselves to bipartite graphs. Basic results on independent dominator sequence number in arbitrary graphs are given in Jayaram et al. [1].

We observe that $l(G) = 1$ if and only if G is isomorphic to $K_{r,s}, with r, s \geq 1$.

For the path P_n we have, $l(P_n) = \lfloor \frac{n}{2} \rfloor$. For the bipartite complement P_n^{bc} of the path $P_n = (v_1, v_2, \ldots, v_n)$, with $n \geq 6$, for any vertex v_i at most two vertices in the other partite set are not dominated by v_i. Thus $l(P_n^{bc}) = 3$.

Proposition 3.3. *For any even cycle C_n we have $l(C_n) = \frac{n}{2} - 1$.*

Proof. Let $C_n = (x_1, x_2, \dots, x_n, x_1)$. Obviously $(x_1, x_3, \dots, x_{n-3})$ is a dominator sequence and hence $l(C_n) \geq \frac{n}{2} - 1$. Further, any dominator sequence of length k in X dominates at least $k+1$ vertices in Y so that $k + 1 \leq \frac{n}{2}$. Hence $k \leq \frac{n}{2} - 1$. Thus $l(C_n) = \frac{n}{2} - 1$. $\square$

Proposition 3.4. *For any even cycle C_n, we have $l(C_n \square K_2) = n - 2$.*

Proof. Let $C_n^0 = (x_1, x_2, \dots, x_n, x_1)$ and $C_n^1 = (y_1, y_2, \dots, y_n, y_1)$ be the two copies of C_n in G with $x_i y_i \in E(G)$. Then $X = \{x_i : 1 \leq i \leq n \text{ and } i \text{ is odd}\} \cup \{y_i : 1 \leq i \leq n \text{ and } i \text{ is even}\}$ and $Y = V(G) - X$ is the bipartition of G. Now $(x_1, y_2, x_3, y_4, \dots, x_{n-3}, y_{n-2})$ is a dominator sequence of G and hence $l(G) \geq n - 2$. Further any dominator sequence of length k in X dominates at least $k+2$ vertices in Y, so that $k+2 \leq n$. Thus $l(G) \leq n-2$ and hence $l(G) = n - 2$. $\square$

Proposition 3.5. *For any path P_n, we have $l(P_n \square K_2) = n - 1$.*

Proof. The proof is similar to that of Proposition 3.4. $\square$

Proposition 3.6. *Let n and k be two positive integers with $n > 2k$. Then there exists a bipartite graph G of order n with $l(G) = k$.*

Proof. Let $A = \{x_1, x_2, \dots, x_k\}, B = \{y_1, y_2, \dots, y_k\}$ and $C = \{z_1, z_2, \dots, z_{n-2k}\}$. Let G be the bipartite graph with bipartition $X = A$ and $Y = B \cup C$, where $N(x_i) = \{y_i\} \cup C, 1 \leq i \leq k$. Clearly $(x_1, x_2, \dots, x_k)$ is a dominator sequence of G and since $|X| = k$, it follows that $l(G) = k$. $\square$

Lemma 3.7. *For any bipartite graph G we have $l(G) \leq \beta_1(G)$, where $\beta_1(G)$ is the cardinality of a maximum matching in G.*

Proof. Let $(x_1, x_2, \dots, x_k)$ be a dominator sequence of G, where $k = l(G)$. For each x_i choose $y_i \in N(x_i) - (N(x_1) \cup N(x_2) \cup \dots \cup N(x_{i-1}))$. Clearly, $x_1y_1, x_2y_2, \dots, x_ky_k$ is a matching in G and hence $l(G) \leq \beta_1(G)$.

$\square$

Lemma 3.8. *Let G be an r-regular bipartite graph of order n. Then $l(G) \leq \frac{n}{2} - r + 1$.*

Proof. Let $(x_1, x_2, \dots, x_k)$ be any dominator sequence of G with $k = l(G)$. Since x_1 dominates exactly r vertices in Y and each x_i dominates at least one vertex in Y not dominated by $x_1, x_2, \dots, x_{i-1}$, it follows that $r + (k - 1) \leq |Y| = \frac{n}{2}$. Hence $l(G) \leq \frac{n}{2} - r + 1$. $\square$

Remark 3.9. *The inequality given in the above theorem can be sharp and also strict. For example, consider the graph in Fig.2(a) with $n = 14$ and $r = 3$. Then $(1, 5, 6, 3, 7)$ is a dominator sequence of length, $l(G) = \frac{n}{2} - r + 1 = 5$. For the graph in Fig.2(b) with $n = 12$ and $r = 3, (1, 2, 4)$ is a sequence of length, $l(G) = 3 < \frac{n}{2} - r + 1$.*

Proposition 3.10. *Let G be any bipartite graph. Then for the shadow graph G_s of G, we have $l(G_s) = 2l(G)$.*

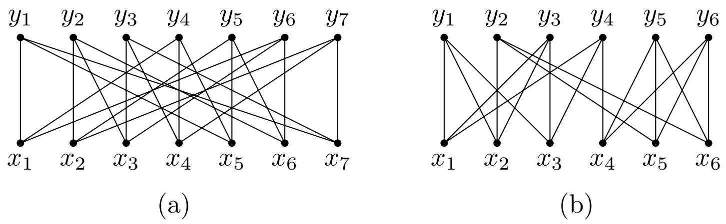


Figure 2: (a)A graph with $l(G) = \frac{n}{2} - r + 1$ (b) A graph with $l(G) < \frac{n}{2} - r + 1$

Proof. Let X, Y be a bipartition of G. Let X' and Y' be the sets of vertices in G_s corresponding to X and Y respectively. Then $X \cup X', Y \cup Y'$ is a bipartition of G_s. Now if $l(G) = k$ and $(x_1, x_2, \ldots, x_k)$ is a dominator sequence of G, then $(x'_1, x'_2, \ldots, x'_k, x_1, x_2, \ldots, x_k)$ is a dominator sequence of G_s. Hence $l(G_s) \geq 2k = 2l(G)$. Now, let S be any dominator sequence of G_s. Since the induced subgraphs $\langle X \cup Y' \rangle$ and $\langle X' \cup Y \rangle$ are both isomorphic to G and the vertices in Y' are dominated only by the vertices in X, it follows that $|S \cap X'| \leq k$ and $|S \cap X| \leq k$. Hence $l(G_s) \leq 2k$. □

Proposition 3.11. *Let G be a connected graph of order n. Then for the subdivision graph $S(G)$ of G, we have $l(S(G)) = n - 1$.*

Proof. The bipartition of $S(G)$ is given by $X = V(G)$ and $Y = V(S(G)) - V(G)$. We consider the dominator sequence to contain vertices from Y. Since any vertex of Y has degree 2 in $S(G)$, any dominator sequence of length $k = l(S(G))$ dominates at least $k + 1$ vertices of $V(G)$ and hence $k + 1 \leq n$. Hence $l(S(G)) \leq n - 1$. Now, let T be any spanning tree of G with $E(T) = \{e_1, e_2, \ldots, e_{n-1}\}$. Let y_i be the vertex in Y that subdivides the edge e_i. Clearly $(y_1, y_2, \ldots, y_{n-1})$ is a dominator sequence of $S(G)$ and hence $l(G) \geq n - 1$. □

We now proceed to obtain a characterization of all bipartite graphs with $l(G) = |X|$. We first define recursively a family $\mathfrak{F}$ of bipartite graphs.

(i) All stars are in $\mathfrak{F}$.

(ii) If $G \in \mathfrak{F}$ and if (X, Y) is a bipartition of G with $|X| \leq |Y|$ and S is any star and G' is the graph obtained from G and S by joining the centre of S to a nonempty subset of Y, then $G' \in \mathfrak{F}$.

Theorem 3.12. *Let G be a connected bipartite graph with bipartition (X, Y) with $|X| \leq |Y|$. Then $l(G) = |X|$ if and only if $G \in \mathfrak{F}$.*

Proof. Suppose $G \in \mathfrak{F}$. Let $X = \{x_1, x_2, \ldots, x_k\}$ be the set of all centres of the stars, in the order in which they appear, in the construction of G. Clearly $X, V(G) - X$ is a bipartition of G and X is a dominator sequence of G. Hence $l(G) = |X|$. Conversely, let G be a bipartite graph with bipartition X, Y such that $X = (x_1, x_2, \ldots, x_k)$ is a dominator sequence of G. Let $S_1 = N[x_1]$ and $S_i = N[x_i] - \bigcup_{j=1}^{i-1} N(x_j)$. Then $\langle S_i \rangle$ is a star with centre x_i and G can be obtained from these stars by using the above construction. Hence $G \in \mathfrak{F}$. □

Definition 3.13. *A bipartite graph G is said to be dominator sequence critical if $l(G) = k$ and $l(G - v) < k$ for all $v \in V(G)$.*

Proposition 3.14. *A bipartite graph G is dominator sequence critical if and only if G has a perfect matching and $l(G) = |X|$.*

Proof. Let X, Y be a bipartition of G and let $l(G) = k$. Let $(x_1, x_2, \dots, x_k)$ be a dominator sequence of G. If G has a perfect matching and $l(G) = |X|$, then $|X| = |Y| = k$ and trivially $l(G - v) < k$ for all $v \in V$. Conversely, suppose $l(G - v) < k$ for all $v \in V$. If $|X| > k$, then $l(G - x) = l(G)$ for any $x \in X - S$, where S is a dominator sequence of G, a contradiction. Thus $|X| = k$. By a similar argument we have $|Y| = k$. Further $\{x_i y_i : y_i \in N(x_i) - \bigcup_{j=1}^{i-1} N(x_j) \text{ and } 1 \leq i \leq k\}$ is a perfect matching in G. □

Definition 3.15. *Let $S = (x_1, x_2, \dots, x_r)$ be a dominator sequence in a bipartite graph G. Let y_i be a vertex which is dominated by x_i but not dominated by $x_1, x_2, \dots, x_{i-1}$. The matching $M = \{x_1y_1, x_2y_2, \dots x_r, y_r\}$ is called a matching determined by S.*

Definition 3.16. *Let $G = (V, E)$ be a graph and M a matching. An M-alternating path in G is a path whose edges are alternatively in $E \backslash M$ and in M. An M-alternating path whose two end vertices are adjacent is called an M-alternating cycle.*

The following result gives us a good characterization to identify a maximum dominator sequence based on the size of an M-alternating cycle in a graph. The following theorem is used later in the paper to identify dominator sequences.

Theorem 3.17. *A maximum matching M in a bipartite graph G is determined by a dominator sequence S in G if and only if there does not exist an M-alternating cycle in G.*

Proof. Suppose there exists a dominator sequence $S = \{x_1, x_2, \dots, x_r\}$ which determines the matching $M = \{x_1y_1, x_2y_2, \dots x_ry_r\}$. Let H be the subgraph of G induced by $V(M) = \{x_1, x_2, \dots, x_r, y_1, y_2, \dots, y_r\}$. Since x_1 dominates y_1, but does not dominate $y_2, y_3, \dots, y_r$ it follows that x_1 has degree 1 in H. Hence the edge x_1y_1 does not lie on an M-alternating cycle in G. Now x_2 is a vertex of degree 1 in $H - \{x_1, y_1\}$ and hence the edge x_2y_2 does not lie in an M-alternating cycle. By repeating this process, we see that no edge of M lies on an M-alternating cycle and hence there does not exist an M-alternating cycle in G.

Conversely, suppose that $M = \{x_1y_1, x_2y_2, \dots x_ry_r\}$ is a maximum matching in G such that there does not exist an M-alternating cycle in G. Let H be the subgraph of G induced by $V(M) = \{x_1, x_2, \dots, x_r, y_1, y_2, \dots, y_r\}$ and let $P = (u = v_1, v_2, \dots, v_s = v)$ be an M-alternating path of maximum length in H. Then the length of P is odd, the first edge v_1v_2 and the last edge $v_{s-1}v_s$ of P are in M and the vertices v_1, v_s are not adjacent to any vertex in $V(H) - V(P)$. Further since there does not exist an M-alternating cycle in G, the vertices v_1, v_s are not adjacent to any vertex in $V(P)$. Thus both v_1 and v_s are pendant vertices in H and since the length of P is odd, one of these vertices, say v_1, is in X. Let $v_1 = x_1, D_1 = \{x_1\}, M_1 = \{x_1y_1\}$ and $H_1 = H - \{x_1, y_1\}$. Now by considering an M-alternating path of maximum length in H_1, we obtain a vertex, say x_2, of degree 1 in $X \cup V(H_1)$. Let $D_2 = \{x_1, x_2\}, M_2 = \{x_1y_1, x_2y_2\}$ and $H_2 = H_1 - \{x_2, y_2\}$. By repeating this process we obtain a dominator sequence $D = \{x_1, x_2, \ldots, x_r\}$ in G which determines the matching M.

$\square$

In the following theorem we give bounds on the number of edges of a bipartite graph G of order n with $l(G) = k$.

Theorem 3.18. *Let G be a bipartite graph of order n and size m with bipartition X, Y, where $r = |X| \leq |Y| = s$ and let $l(G) = k$. Then $n - 1 \leq m \leq rs - \binom{k}{2}$. Further, $m = n - 1$ if and only if G is a tree with $\beta_1(G) = k$. Also, $m = rs - \binom{k}{2}$ if and only if G is isomorphic to the graph G_1 obtained from the complete bipartite graph $K_{r,s}$ with bipartition $X = \{x_1, x_2, \ldots, x_r\}$ and $Y = \{y_1, y_2, \ldots, y_s\}$ by deleting all edges $x_i y_j$ where $1 \leq i < j \leq k$.*

Proof. Let $(x_1, x_2, \ldots, x_k)$ be a dominator sequence in G and let $y_i \in N(x_i) - \bigcup\limits_{j=1}^{i-1} N(x_j), 1 \leq i \leq k$. Then $x_i y_j \notin E(G)$ for all pairs i, j with $1 \leq i < j \leq k$. Hence $m \leq rs - \binom{k}{2}$. Since G is connected, the lower bound is trivial. Now suppose $l(G) = k$ and $m = n - 1$. Then it follows from Theorem 5.1 that G is a tree and $\beta_1(G) = k$. Conversely, if G is a tree with $\beta_1(G) = k$ then trivially $m = n - 1$.

Now let $l(G) = k$ and $m = rs - \binom{k}{2}$. Let $(x_1, x_2, \ldots, x_k)$ be a maximum dominator sequence and let $y_i \in N(x_i) - \bigcup_{j=1}^{i-1} N(x_j)$. Clearly x_i is not adjacent to $y_{i+1}, y_{i+2}, \ldots, y_k, 1 \leq i < k$ and since $m = rs - \binom{k}{2}$ it follows that $G = G_1$. Conversely, suppose G is isomorphic to the graph G_1 given in the theorem. Clearly, $m = rs - \binom{k}{2}$. Also, $(x_1, x_2, \ldots, x_k)$ is a dominator sequence of G_1 and hence $l(G) \geq k$. Further since $N(x_i) = Y$ for all $i \geq k$, it follows that $l(G_1) = k$. $\square$

4 Complexity Results

In this section we prove that the decision problem corresponding to the dominator sequence number is NP-Complete.
DOMINATOR SEQUENCE PROBLEM(DSP)
INSTANCE: A bipartite graph G with bipartition X, Y and a positive integer k.
QUESTION: Is there a sequence $x_{i_1}, x_{i_2}, \ldots, x_{i_k}$ of vertices in X such that $N(x_{i_j}) \setminus (N(x_{i_1}) \cup N(x_{i_2}) \cup \cdots \cup N(x_{i_{j-1}})) \neq \emptyset$ for all $2 \leq j \leq k$?

For this purpose we use the feedback vertex set problem.
FEEDBACK VERTEX SET
INSTANCE: Graph $G = (V, A)$ and a positive integer $k \leq |V|$.
QUESTION: Is there a subset $V' \subseteq V$ with $|V'| \leq k$ such that V' contains at least one vertex from every cycle in G?

Karp [8] has proved that FEEDBACK VERTEX SET is NP-complete. We now define another problem called closure problem.

Definition 4.1. *Let X be a set and let $\mathcal{C}$ be a collection of subsets of X. For a subset $A \subseteq X$ the closure of A, denoted by $\sigma(A)$ is the minimal subset B of X satisfying the following properties:*

1 $A \subseteq B$.

2 *There is no set* $P \in \mathcal{C}$ *such that* $|P \setminus B| = 1$.

Observation 4.2. *Let* $\mathcal{F} = \{B \subseteq X : B \supseteq A$ *and there does not exist* $P \in \mathcal{C}$ *such that* $|P \backslash B| = 1\}$. *Clearly* $X \in \mathcal{F}$ *and if* $B_1, B_2 \in \mathcal{F}$, *then* $B_1 \cap B_2 \in \mathcal{F}$. *Hence the closure of* A *is given by* $\sigma(A) = \bigcap_{B \in \mathcal{F}} B$, *so that* $\sigma(A)$ *is well defined.*

CLOSURE PROBLEM(CP)
INSTANCE: A set X, a collection C of subsets of X and a positive integer k.
QUESTION: Is there a subset $A \subseteq X$ with $|A| \leq k$ and $\sigma(A) = X$.

We first prove that the CLOSURE PROBLEM is NP-complete by reduction from feedback vertex set. We finally prove that dominator sequence problem is NP-complete by reduction from closure problem.

Theorem 4.3. *CLOSURE PROBLEM is NP-Complete.*

Proof. The proof is by reduction from FEEDBACK VERTEX SET PROBLEM. Let (G, k) be an instance of the feedback vertex set problem. We may assume that the minimum degree of G is at least two. With each vertex v of G we associate an element x_v and with each neighbor u of v we associate two elements x^1_{vu}, x^2_{vu}. Note that the element x^1_{vu} is different from the element x^1_{uv}.

Let $X = \{x_v : v \in V\} \cup \{x^1_{vu} : v \in V, u \in N(v)\} \cup \{x^2_{vu} : v \in V, u \in N(v)\}$. Clearly $|X| = n + 4m$ where $n = |V(G)|$ and $m = |E(G)|$. The sets in $\mathcal{C}$ are constructed as follows. For every neighbor u of v, there are three sets in $\mathcal{C}, \{x_w | w \in N(v) \setminus \{u\}\} \cup \{x^1_{vu}\}, \{x_w | w \in N(v) \setminus \{u\}\} \cup \{x^2_{vu}\}$, and $\{x_v, x^1_{vu}, x^2_{vu}\}$. Thus the total number of sets is $6m$. Let $X_V = \{x_v | v \in V(G)\}$.

We claim that G has a feedback vertex set of size at most k if and only if there exists a subset $A \subseteq X$ with $|A| \leq k$ and $\sigma(A) = X$. First suppose F is a feedback vertex set of size at most k. Let $A = \{x_v | v \in F\}$. We claim that $\sigma(A) = X$. We first claim that $X_V \subseteq \sigma(A)$. Suppose $V' = \{v | x_v \notin \sigma(A)\}$ is not empty. Since V' induces a forest in G, there exists a vertex $v \in V'$ which has at most one neighbor in V'. Let u be a neighbor of v such that all the neighbors of v are not in V'. Then $\{x_w | w \in N(v) \setminus \{u\}\} \subseteq \sigma(A)$. Then this implies $x^1_{vu}, x^2_{vu} \in \sigma(A)$ and the set $\{x_v, x^1_{vu}, x^2_{vu}\} \subseteq A$, which implies that $x_v \in \sigma(A)$, a contradiction. If $X_V \subseteq \sigma(A)$, then it follows that all the elements x^1_{vu}, x^2_{vu} must also be in $\sigma(A)$, and hence $\sigma(A) = X$.

Now let $A \subseteq X$ be a subset such that $\sigma(A) = X$ and $|A| \leq k$. Choose such a set A such that $|A \setminus X_V|$ is minimum. Suppose $x^1_{vu} \in A$ for some vertex v and neighbor u of v. By the choice of $A, A' = \sigma(A \setminus \{x^1_{vu}\}) \subset X$. Note that A' cannot contain X_V otherwise $A' = X$. Since $\sigma(A' \cup \{x^1_{vu}\}) = X$, there must be a set B such that $|B \setminus (A' \cup \{x^1_{vu}\})| = 1$. Thus the set B must contain x^1_{vu}, some element $x \notin A' \cup \{x^1_{vu}\}$, and all other elements in B are in A'. If $x = x_w$ for some $w \in V(G)$, let $A_1 = (A \cup \{x_w\}) \setminus \{x^1_{vu}\}$. Then $\sigma(A_1) = X$ and A_1 contains fewer elements not in X_V, contradicting the choice of A. Therefore $x = x^i_{pq}$ for some vertex p and neighbor q of p, and $i \in \{1, 2\}$. But there is a set that contains x^1_{vu} and x^i_{pq} iff $v = p, u = q$ and $i = 2$. The only such set is $\{x_v, x^1_{vu}, x^2_{vu}\}$. This implies that $x_v \in A'$. Again since $\sigma(A \cup \{x^1_{vu}\} \cup \{x^2_{vu}\}) = X$, there is a set B_1 such that $|B_1 \setminus (A' \cup \{x_{vu}^1\} \cup \{x_{vu}^2\})| = 1, B_1$ contains exactly one of x_{vu}^1 or x_{vu}^2 and other elements in B_1 are in A'. Then the element in $B_1 \setminus (A' \cup \{x_{vu}^1\} \cup \{x_{vu}^2\})$ must be x_w for some vertex w. Now replace x_{vu}^1 in A by x_w, to get a set containing fewer elements not in X_V, but whose closure is X, contradicting the choice of A.

So, we may assume $A \subseteq X_V$. We claim that $F = \{v | x_v \in A\}$ is a feedback vertex set in G. Let $V' = V(G) \setminus F$ and suppose there is a cycle C in the subgraph induced by V'. We show that the closure of A is a proper subset of X, a contradiction.

Consider the set $B' = \{x_v | v \in V(C)\} \cup \{x_{vu}^1, x_{vu}^2 | (N(v) \setminus \{u\}) \cap V(C) \neq \emptyset, uv \in E(G)\}$ and let $B = X \setminus B'$. Then B contains A and is a proper subset of X. We show that $\sigma(B) = B$, giving a contradiction. To do that, it is sufficient to show that any set that contains an element of B' contains at least two such elements.

Suppose some set contains an element $x_v \in B'$. The sets containing x_v are of the form $\{x_v, x_{vu}^1, x_{vu}^2\}$ for some neighbor u of v, or $\{x_p | p \in N(u) \setminus \{w\}\} \cup \{x_{uw}^i\}$ for some neighbor u of v, a neighbor w of $u \neq v$, and $i \in \{1, 2\}$. In the first case, both $x_{vu}^1, x_{vu}^2 \in B'$ as v has a neighbor other than u in the cycle. Similarly, $x_{uw}^i \in B'$ since u has a neighbor v other than w in the cycle. Suppose B' contains an element x_{vu}^i. Then v has a neighbor w other than u in C. The sets containing x_{vu}^i are $\{x_v, x_{vu}^1, x_{vu}^2\}$ and $\{x_p | p \in N(v) \setminus \{u\}\} \cup \{x_{vu}^i\}$. In the first case, both x_{vu}^1, x_{vu}^2 must be in B' and in the second case x_w is in B' since $w \in V(C)$ and $w \neq u$.

Therefore B satisfies the required properties and $\sigma(B) = B$. This contradicts $\sigma(A) = X$. Thus F must be a feedback vertex set of size at most k. □

Theorem 4.4. *DOMINATOR SEQUENCE PROBLEM(DSP) is NP-Complete.*

Proof. The reduction is from CLOSURE PROBLEM. Consider an instance of the closure problem $X, \mathcal{C} = \{C_1, C_2, \ldots, C_l\} \cup \{X\}$ and $|X| - k$. Let G be the bipartite graph with bipartion X, Y, where $|Y| = l$ and $\{N(y_i) : y_i \in Y\} = C_i, 1 \leq i \leq l$. Let $A' = (x_{i_1}, x_{i_2}, \ldots, x_{i_k})$ be a sequence of vertices in X, and $A = X \setminus A'$ such that the closure $\sigma(A) = X$. We claim that $N(x_{i_j}) \setminus (N(x_{i_1}) \cup N(x_{i_2}) \cup \cdots \cup N(x_{i_{j-1}})) \neq \emptyset$ for all $2 \leq j \leq k$. We note that $|C_i \setminus X| = 0, 1 \leq i \leq l$, since the closure of A is the entire set X. Suppose on the contrary, let $x_{i_j} \in A'$ be a vertex that does not satisfy the property. Then, deleting x_{i_j} from X, we note that for any C_i, if $x_{i_j} \notin C_i$ then $|C_i \setminus X| = 0$, contradicting the minimality of X and if $x_{i_j} \in C_i$, then removing x_{i_j} from X, also removes x_{i_j} from C_i, and hence $|C_i \setminus X| = 0$, again contradicting the minimality of X. Hence $\{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\}$ is a dominator sequence of G. Conversely, suppose G has a dominator sequence $(x_{i_1}, x_{i_2}, \ldots, x_{i_k})$. Then $N(x_{i_j}) \setminus (N(x_{i_1}) \cup N(x_{i_2}) \cup \cdots \cup N(x_{i_{j-1}})) \neq \emptyset$ for all $2 \leq j \leq k$. Let $A' = \{x_{i_j} | 1 \leq j \leq k\}$ and let $A = X \setminus A'$. Then we claim that the closure of A is X. Suppose not and let j be the largest number $\leq k$ such that $x_{i_j} \notin \sigma(A)$. If $j = 1$, then $|X \setminus \sigma(A)| = 1$ and since $X \in \mathcal{C}$, this contradicts the definition of $\sigma(A)$. Suppose $j \geq 2$ and let y_{i_j} be an element in $N(x_{i_j}) \setminus N(x_{i_1}) \cup N(x_{i_2}) \cup \cdots \cup N(x_{i_{j-1}})$. Then $N(y_{i_j})$ is a set in $\mathcal{C}$ such that $|N(y_{i_j}) \setminus \sigma(A)| = 1$, a contradiction. Thus $\sigma(A) = X$. □

5 Dominator Sequences in Trees

In this section we present a linear time algorithm to obtain a maximum dominator sequence for trees. It follows from Theorem 3.17 that for any tree T, we have $l(T) = \beta_1(T)$.

Theorem 5.1. *For any tree T, a dominator sequence D of T with $|D| = l(T) = \beta_1(T)$ can be determined in linear time.*

Proof. Let X, Y be the bipartition of T. A maximum matching M of T with $|M| = \beta_1(T)$ can be obtained in linear time(See problem 7.95 in[10]). Let T be a tree and let M be a maximum matching in T. Let T' be a BFS tree of T rooted at a vertex r. For each vertex $v \in V(T)$, let $h(v)$ denote the level of v in T'. Let $V(M)$ be the vertex induced subgraph of M in T'. Clearly $V(M)$ is a forest and $\forall v \in V(M), v$ is a matched vertex. Let $T_0 = V(M), D = \emptyset$ and h_l be the highest level such that the vertices at level h_l belong to the X partition of T'. For each pendant vertex v at level h_l, let $D = D \cup \{v\}, T' = T' - \{uv\}$, where u is a matched vertex with v at level $h_l - 1$. Since the set of all vertices at level h_l are independent and $V(M)$ is a vertex induced matching, each vertex v at level h_l dominates a new vertex at level $h_l - 1$. Again, for each pendant vertex v at level $h_l - 2$, let $D = D \cup \{v\}, T' = T' - \{uv\}$. We repeat this process for all alternate levels until either level 1 or level 2 is reached and repeat the same process starting from level 1 or level 2 until level h_l is reached or until there are no more matching edges remaining. Clearly, D is a dominator sequence and since we process all edges of the tree atmost once, it is easy to see that D can be obtained in $O(n)$ time. $\square$

The following algorithm gives a maximum dominator sequence for any given tree, T in linear time.

Algorithm : Tree Dominator Sequence(TDS)
Input: A tree $T(X, Y)$ with $|V(T)| = n$.
Output: A maximum dominator sequence $L(T) = (x_1, x_2, \ldots, x_k)$.

1. Initialize $L(T) = \phi$, X' to be an empty stack. Select a vertex r as root vertex. Assign levels to all the vertices in T with the level of r as zero. Let l be the highest level.

2. Compute a maximum matching $M = \{x_1y_1, x_2y_2, \ldots, x_ky_k\}$ of T, where $k = \beta_1(T)$.

3. For level $j = l, l-1, l-2, \ldots, , 0$ do

    3.1. For each x_i at level j

    3.1.1. If $deg(x_i) = 1$, then append x_i to $L(T)$. Assign UNIQUE$(x_i) = y_i$ where $y_i \in N(x_i)$ and remove x_i and y_i.

    3.1.2. If there exists a pendant vertex $v \in N(x_i)$, then add x_i to X', assign UNIQUE$(x_i) = v$ and remove x_i and v.

4. For each element x_i on top of X' append x_i to $L(T)$.

5. Output $L(T)$.

Complexity: Assigning levels to each vertex can be done through a breadth first search of the tree. An $O(n)$ linear time algorithm for computing a maximum matching in trees is known(See problem 7.95 in [10]). In step 3, each vertex of the partite set X in the matching is processed and hence the complexity is the sum of the degrees, which in the case of trees is $O(n)$. Hence, the complexity of finding a maximum dominator sequence in trees is $O(n)$. We need the following properties that arise from the breadth first search T used to assign levels to the vertices.

P1: A vertex $v \neq r$ at level j is adjacent to exactly one vertex at level $j-1$.

P2: No vertex v at level j is adjacent to any vertex at any level higher than $j+1$.

P3: All vertices on a given level form an independent set.

In the following, we shall refer to vertices identified in step 3.1.1 as *type*-1 vertices and vertices identified in step 3.1.2 as *type*-2 vertices.

Theorem 5.2. *The sequence $L(T)$ given by the algorithm TDS is a maximum dominator sequence of T.*

Proof. We prove by showing that UNIQUE(x_i) is the vertex adjacent to x_i and not adjacent to any x_j that occurs earlier to x_i in the sequence $L(T)$. It follows from $P1$ that if x_i and x_j are of *type*-1 and are at the same level, then UNIQUE(x_i) is not adjacent to x_j and UNIQUE(x_j) is not adjacent to x_i. Now, suppose x_i and x_j that are of *type*-2. If x_i and x_j are at the same level then, UNIQUE(x_i) is not adjacent to x_j and UNIQUE(x_j) is not adjacent to x_i. If x_j and x_i are at different levels and the level of x_j is lower than the level of x_i, then step 4 of the algorithm TDS ensures that x_j occurs before x_i in $L(T)$. In this case x_j is at least two levels lower than UNIQUE(x_i), and hence by $P2$, x_j is not adjacent to UNIQUE(x_i).

If x_i is of *type*-1 and x_j is of *type*-2, then x_i occurs before x_j in $L(T)$ and it follows from $P1, P2$ and $P3$ that UNIQUE(x_j) is not adjacent to x_i.

Since all x_i's are in $L(T)$ and $|L(T)| = |M| = \beta_1(T)$, $L(T)$ is a maximum dominator sequence. □

6 Conclusion

Motivated by an application in the design of survivable logical topology mapping (SLTM) in an IP-over-WDM optical network, we defined the concept of dominator sequence in a bipartite graph and initiated a study of the corresponding parameter.

The concept of dominator sequence can be extended to general graphs. For instance, we can define a set $X = \{x_1, \ldots, x_k\}$ to be a dominator sequence of a graph $G = (V, E)$ if X is an independent set and each $x_i \in X$ is adjacent to at least one vertex not dominated by $x_1, x_2, \ldots, x_{i-1}$.

Another definition of dominator sequence for general graphs is to drop the requirement that the set X be independent. In fact, such a sequence called, *incident cover sequence*, was defined in [13] and [2] and was used in [15] to study the problem of augmenting the logical graph with additional edges to guarantee that the augmented graph admits a survivable mapping.

We believe that further study of the dominator sequence concept for general and bipartite graphs will have significant theoretical as well as practical implications for the survivable logical topology mapping problem.

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References

[1] S. Arumugam, B. Jayaram and K. Thulasiraman, Independent Dominator Sequence Number of a Graph, *Procedia Computer Science*, **74** (2015), 43–46.

[2] B. Brešar, T. Gologranc, M. Milanič and D.F. Rall, Dominating sequence in graphs, *Discrete Math.*, **336** (2014), 22–36.

[3] B. Brešar, S. Klavzar and D.F. Rall, Domination game and the imagination strategy, *SIAM J. Discrete Math.*, **24** (2000), 979–991.

[4] G. Chartrand and L. Lesniak, *Graphs and Digraphs*, (5th ed.) Belmont, CA, USA: Wadsworth Publ. Co., 1986.

[5] I. Chlamtac, A. Ganz and G. Karmi, Lightpath Communications: An Approach to High Bandwidth Optical WDM, *IEEE Transactions on Communications*, **40**(7) (1992), 1171–1182.

[6] S. Hedetniemi and R. Laskar, A bipartite theory of graphs: I, *Congr. Numer.*, **55** (1986), 5–14.

[7] S. Hedetniemi and R. Laskar, A bipartite theory of graphs: II, *Congr. Numer.*, **64** (1988), 137–146.

[8] R.M. Karp, Reducibility among combinatorial problems, in R.E. Miller and J.W. Thatcher (eds), Complexity of Computer Computations, Plenum Press, New York, (1972), 85–103.

[9] M. Kurant and P. Thiran, *Survivable mapping algorithm by ring trimming(SMART) for large IP-over-WDM networks*, BroadNets, 2004.

[10] U. Manber, *Introduction to algorithms: a creative approach*, Addison-Wesley, 1989.

[11] M.N.S.Swamy and K. Thulasiraman, *Graphs, Networks and Algorithms*, Wiley Interscience, 1981.

[12] E. Modiano and A. Narula-Tam, Survivable Lightpath Routing: A New Approach to the Design of WDM-Based Networks, *IEEE Journal on Selected Areas in Communications*, **20** (2002), 800–809.

[13] K. Thulasiraman, M. Javed and G. Xue, *Circuits/Cutsets Duality and a Unified Algorithmic Framework for Survivable Logical Topology Design in IP-over-WDM Optical Networks*, INFOCOM 2009, (2009), 1026–1034.

[14] K. Thulasiraman, M. Javed and G. Xue, *Primal meets Dual: A generalized theory of logical topology survivability in IP-over-WDM optical networks*, Proceedings of the 2nd international conference on COMmunication systems and NETworks, COMSNETS'10, IEEE Press, Piscataway, NJ, USA, (2010), 128–137.

[15] K. Thulasiraman, T. Lin, M. Javed and G. Xue, Logical topology augmentation for guaranteed survivability under multiple failures in IP-over-WDM optical networks, *Optical Switching and Networking*, (2010), 206–214.

[16] K. Thulasiraman, T. Lin and Z. Zhou, *Robustness of Logical Topology Mapping Algorithms for Survivability against Multiple Failures in an IP-Over-WDM Optical Network*, Proceedings of the international conference on COMmunication systems and NETworks, COMSNETS'12, IEEE Press, Piscataway, NJ, USA, (2012), 1–9.

[17] D.B. West, *Introduction to Graph Theory*, Prentice Hall, 1996.