# Conditional Diagnosability of Matching Composition Networks Under the PMC Model

Min Xu, Krishnaiyan Thulasiraman, and Xiao-Dong Hu

***Abstract*—In the work of Lai *et al.* in 2005, they proposed a new measure for fault diagnosis of systems, namely, *conditional diagnosability*. It assumes that no fault set can contain all the neighbors of any vertex in the system. In the same paper, they showed that the conditional diagnosability of hypercube $Q_n$ is $4(n-2)+1$ for $n \geq 5$. In this brief, we generalize this result by considering a family of more popular networks, namely, matching composition networks (MCNs), which are a class of networks composed of two components of the same order linked by a perfect matching under PMC (Preparata, Metze and Chien) model. We determine in Theorem 7 the conditional diagnosability for some MCNs, from which we deduce that the hypercube $Q_n$, the crossed cube $\mathrm{CQ}_n$, the twisted cube $\mathrm{TQ}_n$, and the Möbius cube $\mathrm{MQ}_n$ all have the same conditional diagnosability of $4(n-2)+1$ for $n \geq 5$. We show that the bijective connection (BC) networks in the work of Fan and He in 2003 and the work of Zhu in 2008 satisfy the conditions of Theorem 7, and thus, our conditional diagnosability result also applies to BC networks. Finally, we show that the MCNs satisfying the conditions of Theorem 7 are more general than the BC networks.**

***Index Terms*—Conditional diagnosability, conditional faulty set, diagnosability, PMC model.**

## I. Introduction

ADVANCES in the semiconductor technology have made possible the development of very large digital systems comprising hundreds of thousands of components or units. Yet, it is almost impossible to build such systems without defects. Testing of such systems becomes extremely difficult due to their large sizes. First, the complexity of test generation for such large systems is overwhelming. Second, the application of test data, and observation and analysis of test responses are extremely difficult and costly, even if test data could be generated. This problem may further be aggravated by possible geographical distribution of units. Testing of such systems with the traditional stimulus-supplying and response-observing philosophy has become virtually impossible. In 1967, Preparata *et al.* [13] proposed a model and a framework, called system-level diagnosis, for dealing with this problem. In the more than four decades following this pioneering work, several issues arising from the application of this framework have been investigated and resolved. Many of these results have profound theoretical and practical implications. Most of the recent research efforts in system-level diagnosis have focused on enhancing the applicability of system-level diagnosis-based approaches to practical scenarios such as VLSI testing [8], online distributed diagnosis [12], diagnosis under local constraints [4], and, more recently, on diagnosis of interconnection networks employed in parallel computers [10], [11]. Several models of diagnosis (e.g., PMC, BGM, and comparison models) have been considered in the system-level diagnosis literature. Among these models, the most popular one is the PMC model proposed by Preparata [13]. There has been extensive research on fault diagnosis under the PMC model. For examples, see [2], [3], and [11].

In the PMC model, all units in the system under diagnosis can test each other. The test outcomes are "faulty" or "fault free." It is assumed that the test outcomes are correct if the testing unit is fault free; otherwise, the outcomes are unreliable. The set of tests can be represented by a directed graph $G=(V,E)$, in which each vertex represents a unit, and an edge $(u,v)$ indicates that the unit $u$ has tested unit $v$. The collection of all outcomes is called the syndrome $\sigma$. $\sigma(u,v)$ denotes the outcome of unit $u$ testing unit $v$. Without loss of generality, it is assumed that $\sigma(u,v)=1$ if the test outcome is faulty; otherwise, it is 0. The diagnosis problem is to use the syndrome and determine the status (faulty or fault free) of each unit in the system.

For a given syndrome $\sigma$, a subset of vertices $F \subseteq V$ is said to be *consistent* with $\sigma$ if the syndrome $\sigma$ can be produced from the situation that, for any $(u,v) \in E$ such that $u \in V-F$, $\sigma(u,v)=1$ iff $v \in F$. This means that $F$ is a possible set of faulty units. Since a test outcome produced by a faulty unit is unreliable, a given set $F$ of faulty vertices may produce a lot of different syndromes. On the other hand, different fault sets may produce the same syndrome. Let $\sigma(F)$ represent the set of all syndromes that could be produced by $F$. Two distinct sets $F_1, F_2 \subset V$ are said to be *indistinguishable* if $\sigma(F_1) \cap \sigma(F_2) \neq \emptyset$; otherwise, $F_1$ and $F_2$ are said to be *distinguishable*. We say that $(F_1, F_2)$ is an *indistinguishable pair* if $\sigma(F_1) \cap \sigma(F_2) \neq \emptyset$; else, $(F_1, F_2)$ is a *distinguishable pair*. If $F_1$ and $F_2$ are two distinct sets, let the symmetric difference $F_1 \triangle F_2 = (F_1 - F_2) \cup (F_2 - F_1)$. The following lemma gives a necessary and sufficient condition for a pair of sets to be distinguishable.

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*Lemma 1 [2]:* For any two distinct sets $F_1, F_2 \subset V$, $(F_1, F_2)$ is a distinguishable pair iff there exists a vertex $u \in V - (F_1 \cup F_2)$ and there exists a vertex $v \in F_1 \triangle F_2$ such that $(u, v) \in E$.

A system is said to be *$t$-diagnosable* if, given a syndrome, all processors can correctly be identified as faulty or faulty free, provided that the number of faulty processors present in the system does not exceed $t$. The *$t$-diagnosability problem* is to determine the largest value of $t$ for which a given system is *$t$-diagnosable*. In applications of fault diagnosis systems, the probability that all neighbors of a vertex are faulty simultaneously is very small. Thus, Lai *et al.* [11] proposed a new measure of diagnosability. A fault set $F \subset V$ is called a *conditional faulty set* if $N(v) \nsubseteq F$ for any vertex $v \in V$, where $N(v)$ is the set of neighbors of $v$. A system $G(V, E)$ is *conditionally $t$-diagnosable* if, given a syndrome, all processors can correctly be identified as faulty or faulty free, provided that the cardinality of the conditional faulty set present in the system does not exceed $t$. In other words, a system $G(V, E)$ is *conditionally $t$-diagnosable* if $F_1$ and $F_2$ are distinguishable, for each pair of conditional faulty sets $F_1, F_2 \subset V$, and $F_1 \neq F_2$, with $|F_1| \leq t$ and $|F_2| \leq t$. The *conditional diagnosability* of a system $G$, which is written as $t_c(G)$, is defined to be the maximum value of $t$ such that $G$ is conditionally $t$-diagnosable. The *matching composition network* (MCN) is a network composed of two components with the same order linked by a perfect matching. Given a positive integer $t$, let $G_1$ and $G_2$ be two networks with $t$ vertices, and let $M$ be any arbitrary perfect matching between the vertices of $G_1$ and $G_2$, that is, a set of $t$ edges with one endpoint in $G_1$ and the other endpoint in $G_2$. An MCN is defined as a network $G(G_1, G_2; M)$ with the vertex set $V(G(G_1, G_2; M)) = V(G_1) \cup V(G_2)$ and the edge set $E(G(G_1, G_2; M)) = E(G_1) \cup E(G_2) \cup M$.

For any two vertices $u$ and $v$ in $V(G)$ of a network $G$, let $C(G; u, v)$ denote the number of vertices who are the neighbors of both $u$ and $v$, that is, $C(G; u, v) = |\{w : w \in N_G(u)$ and $w \in N_G(v)\}|$, and $C(G) \equiv \max\{C(G; u, v) : u, v \in V(G)\}$. For a connected network $G(V, E)$, if $G - S$ is still connected for any $S \subseteq V(G)$ with $|S| \leq k - 1$, then $G$ is $k$-connected. For a vertex $u \in V(G)$, we use the symbol $N_G(u)$ to denote a set of vertices adjacent to $u$. For a vertex set $U \subseteq V(G)$, let $N_G(U) = \cup_{u \in U} N_G(u) - U$. If $|N_G(u)| = k$ for any vertex in $G$, then $G$ is $k$-regular. If $N_G(u) \cap N_G(v) = \emptyset$ for any edge $(u, v)$ in $G$, then $G$ is triangle free.

Lai *et al.* [11] also showed that the conditional diagnosability of hypercube is $4(n-2)+1$ for $n \geq 5$. In this brief, we generalize this result by considering a family of more popular networks, namely, the MCNs. We will determine in Theorem 7 the conditional diagnosability of a special class of MCNs $G(G_1, G_2; M)$, where $G_1$ and $G_2$ are $t$-regular $t$-connected triangle-free networks, and $C(G) = 2$. As an application, we obtain the conditional diagnosability of many famous networks, such as hypercube [14], crossed cubes [5], Möbius cubes [1], twisted cubes [7], etc. We also show that the bijective connection (BC) networks [6], [16] satisfy the conditions of Theorem 7, and thus, our conditional diagnosability result also applies to BC networks. Finally, we show that the MCNs satisfying the conditions of Theorem 7 are more general than the BC networks.

## II. Conditional Diagnosability of a Class of MCNs

First, we will give some lemmas about MCNs that will be used in the proof of the main result.

*Lemma 2:* Suppose $G = (G_1, G_2; M)$ is an MCN, where $G_1$ and $G_2$ are $(n-1)$-regular $(n-1)$-connected triangle-free networks satisfying $C(G_1) \leq 2$ and $C(G_2) \leq 2$. Then, $G$ is an $n$-regular $n$-connected triangle-free network with $C(G) \leq 2$.

*Proof:* Notice that $G$ is an $n$-connected $n$-regular triangle-free network with $C(G) \leq 2$ and $|V(G_1)| = |V(G_2)| \geq 2(n-1)$.

Next, we will prove that $G$ is $n$-connected. Let $S$ be an arbitrary vertex set of $G$ such that $|S| \leq n-1$. Let $S_1 = S \cap V(G_1)$ and $S_2 = S \cap V(G_2)$. We consider two cases.

**Case 1**: $|S_1| \leq n-2$ and $|S_2| \leq n-2$. In this case, both $G_1 - S_1$ and $G_2 - S_2$ are connected. Since $|V(G_1 - S_1)| \geq 2(n-1) - (n-2) \geq n-1 > |S_2|$, $G - S$ is connected.

**Case 2**: $|S_1| = 0$ and $|S_2| \leq n-1$ or $|S_1| \leq n-1$ and $|S_2| = 0$. Assume without loss of generality that $|S_1| = 0$ and $|S_2| \leq n-1$. Then, $G - S_2$ is still connected. ■

*Lemma 3:* Suppose that $G$ is an $n$-regular triangle-free connected network $G$ with $C(G) = 2$, then $t_c(G) \leq 4(n-2)+1$ for $n \geq 2$.

*Proof:* Since $G$ is a triangle-free connected network with $C(G) = 2$, then there exists the shortest cycle $C = \langle u_1, u_2, u_3, u_4, u_1 \rangle$ of length four in $G$. Let $S = N_G(C) = N(u_1) \cup N(u_2) \cup N(u_3) \cup N(u_4) \backslash \{u_1, u_2, u_3, u_4\}$. Then, $|S| = 4(n-2)$. The sets $F_1 = S \cup \{u_1, u_2\}$ and $F_2 = S \cup \{u_3, u_4\}$ are conditional faulty sets with order $4(n-2)+2$. There is no edge between $V - F_1 - F_2$ and $\{u_1, u_2, u_3, u_4\}$. Then, by Lemma 1, $F_1$ and $F_2$ are indistinguishable. Thus, we have $t_c(G) \leq 4(n-2)+1$ for $n \geq 2$. ■

*Lemma 4 [11]:* Let $G(V, E)$ be a system and $(F_1, F_2)$ be an indistinguishable conditional pair with $F_1 \neq F_2$, then the following two conditions hold:

1) $|N(u) \cap (V - (F_1 \cup F_2))| \geq 1$ for $u \in (V - (F_1 \cup F_2))$;
2) $|N(v) \cap (F_1 - F_2)| \geq 1$ and $|N(v) \cap (F_2 - F_1)| \geq 1$ for $v \in F_1 \triangle F_2$.

*Lemma 5:* Suppose $G$ is an $n$-regular connected triangle-free network with $C(G) \leq 2$. Let $F_1, F_2 \subset V(G)$, $F_1 \neq F_2$, be an distinguishable conditional pair, and let $C$ be a connected component in $(G - F_1 \cap F_2) \cap (F_1 \triangle F_2)$, then three conditions hold.

1) There exists a cycle in $C$ with length at least four.
2) There exists a path $P_4$ of length four in $C$, and $|N_G(P_4)| \geq 4n - 10$.
3) If $|V(C)| \leq 6$, the length of the shortest cycle $\tilde{C}$ in $C$ is four, and $|N_G(\tilde{C})| = 4(n-2)$.

*Proof:* By Lemma 4, all vertices in $C$ have a degree greater than one. Then, there exists a cycle in $C$. In addition, since $G$ is a triangle-free network, then $|V(C)| \geq 4$. Condition 1 holds.

Since the length of cycle in $C$ is at least four, there exists a path $P_4 = \langle x_1, x_2, x_3, x_4 \rangle$ of length four such that $\{(x_1, x_2), (x_2, x_3), (x_3, x_4)\} \subseteq E(C)$. Since $G$ is triangle free and $C(G) \leq 2$, if $(x_1, x_4) \in E(C)$, then $|N_G(P_4)| = 4(n-2) \geq 4n - 10$. If $(x_1, x_4) \notin E(C)$, $x_1$ has a neighbor set $X_1$

with order $(n-1)$ in $V(G)-V(P_4)$; $x_2$ has a neighbor set $X_2$ with order $(n-2)$ in $V(G)-V(P_4)-X_1$; $x_3$ has a neighbor set $X_3$ with order at least $(n-3)$ in $V(G)-V(P_4)-X_1-X_2$; and $x_4$ has a neighbor set $X_4$ with order at least $(n-4)$ in $V(G)-V(P_4)-X_1-X_2-X_3$. We have $|N_G(P_4)|=|X_1|+|X_2|+|X_3|+|X_4|\geq 4n-10$.

Thus, Condition 2 holds. Next, we will prove that Condition 3 holds by the contradiction argument. Let $\tilde{C}$ be a cycle with the shortest length in $C$.

If $|V(\tilde{C})|=6$, then $C=\tilde{C}$. Let $\tilde{C}=\langle x_1,x_2,x_3,x_4,x_5,x_6,x_1\rangle$, where $\{(x_1,x_2),(x_2,x_3),(x_3,x_4),(x_4,x_5),(x_5,x_6),(x_6,x_1)\}\subseteq E(G)$. Assume without loss of generality that $x_1\in F_1-F_2$. By Lemma 4, $x_2\in F_2-F_1$, $x_6\in F_1-F_2$, or $x_2\in F_1-F_2$, $x_6\in F_2-F_1$. Assume without loss of generality that $x_2\in F_1-F_2$ and $x_6\in F_2-F_1$. Then, by Lemma 4, $x_3\in F_2-F_1$, $x_4\in F_2-F_1$, $x_5\in F_1-F_2$, and $x_6\in F_1-F_2$, which contradicts with $x_6\in F_2-F_1$. Then, $|V(\tilde{C})|<6$.

If $|V(\tilde{C})|=5$, then $C=\tilde{C}$. Otherwise, there exists a vertex $u\in V(C)$, but $u\notin V(\tilde{C})$, then we can construct a cycle of length less than five in $C$ by Lemma 4, which is a contradiction. Let $\tilde{C}=\langle x_1,x_2,x_3,x_4,x_5,x_1\rangle$, where $\{(x_1,x_2),(x_2,x_3),(x_3,x_4),(x_4,x_5),(x_5,x_1)\}\subseteq E(G)$. Assume without loss of generality that $x_1\in F_1-F_2$. By Lemma 4, $x_2\in F_2-F_1$ and $x_5\in F_1-F_2$, or $x_2\in F_1-F_2$ and $x_5\in F_2-F_1$. Assume without loss of generality that $x_2\in F_1-F_2$ and $x_5\in F_2-F_1$. Then, by Lemma 4, $x_3\in F_2-F_1$, $x_4\in F_2-F_1$, and $x_5\in F_1-F_2$, which contradicts with $x_5\in F_2-F_1$. Then, $|V(\tilde{C})|<5$.

From the aforementioned discussion, we have $|V(\tilde{C})|=4$ and $|N_G(\tilde{C})|=4(n-2)$. Condition 3 holds. ■

*Lemma 6:* Suppose $G=(G_1,G_2;M)$ is an MCN, where $G_1$ and $G_2$ are $(n-1)$-regular $(n-1)$-connected triangle-free networks with order no less than $4(n-2)+2$, and $C(G_1)\leq 2$ and $C(G_2)\leq 2$. Let $F_1,F_2\subset V(G)$, $F_1\neq F_2$, be an distinguishable conditional pair. Then, either $|F_1|\geq 4(n-2)+2$ or $|F_2|\geq 4(n-2)+2$ for $n\geq 5$.

*Proof:* Since $F_1$ and $F_2$ are distinguishable, then there exists no edge between $G-(F_1\cup F_2)$ and $F_1\triangle F_2$. Let $S=F_1\cap F_2$ and choose a component $C$ in $(G-S)\cap(F_1\triangle F_2)$ if $G-S$ is disconnected; otherwise, let $C=F_1\triangle F_2$. By Lemma 4, we have $|V(C)|\geq 4$ since $G$ is triangle free. Thus, we only need to prove that $|S|+\lceil|V(C)|/2\rceil\geq 4(n-2)+2$.

Let $S_1=V(G_1)\cap S$ and $S_2=V(G_2)\cap S$. We will prove the result by two cases.

**Case 1**: $V(C)\subseteq V(G_1-S_1)$ or $V(C)\subseteq V(G_2-S_2)$.

Assume without loss of generality that $V(C)\subseteq V(G_1-S_1)$. As $V(C)\subseteq V(G_1-S_1)$, we have $N_{G_2}(C)\subseteq S_2$. Thus, $|S_2|\geq|V(C)|$. We now consider two subcases.

**Subcase 1.1**: $G_1-S_1$ is connected. In this case, we have $4(n-2)+2\leq|V(G_1)|=|S_1|+|V(C)|\leq|S_1|+|S_2|=|S|$. Hence, $|S|+\lceil|V(C)|/2\rceil\geq 4(n-2)+2$.

**Subcase 1.2**: $G_1-S_1$ is disconnected. By Lemma 4, the degree of every vertex in $C$ is greater than one. Then, there exists a cycle in $C$.

If $|V(C)|\leq 6$, then, from Lemma 5, we deduce that the length of the shortest cycle $\tilde{C}$ in $C$ is four. Thus, $|N_G(\tilde{C})|=4(n-2)$ and $|N_{G_1}(\tilde{C})|=4(n-3)$, where $N_G(\tilde{C})\subseteq V(G-\tilde{C})$ and $N_{G_1}(\tilde{C})\subseteq V(G_1-\tilde{C})\subset V(G)$. Since $G_1-S_1$ is disconnected, $N_{G_1}(\tilde{C})\subseteq S_1\cup V(C-\tilde{C})$. We have

$$\begin{aligned}|S|+\left\lceil\frac{|V(C)|}{2}\right\rceil&=|S_1|+|S_2|+\left\lceil\frac{|V(C)|}{2}\right\rceil\\&\geq|S_1|+|V(C)|+\left\lceil\frac{|V(C)|}{2}\right\rceil\\&\geq|S_1|+\left|V(C-\tilde{C})\right|+\left|V(\tilde{C})\right|+\left\lceil\frac{|V(C)|}{2}\right\rceil\\&\geq 4(n-3)+4+2\\&\geq 4(n-2)+2.\end{aligned}$$

If $|V(C)|\geq 7$, then, by Lemma 5, there exists a path $P_4$ of length four that has $4n-10$ neighbors in $V(G-P_4)$ and $4n-14$ neighbors in $V(G_1-P_4)$. Since $G_1-S_1$ is disconnected, $N_{G_1}(P_4)\subseteq S_1\cup V(C-P_4)$. Then, $|S_1|+|V(C)|\geq(4n-14)+4=4n-10$. Thus, we have

$$\begin{aligned}|S|+\left\lceil\frac{|V(C)|}{2}\right\rceil&=|S_1|+|S_2|+\left\lceil\frac{|V(C)|}{2}\right\rceil\\&\geq|S_1|+|V(C)|+\left\lceil\frac{|V(C)|}{2}\right\rceil\\&\geq(4n-10)+4\\&\geq 4(n-2)+2.\end{aligned}$$

**Case 2**: $V(C)\cap V(G_1-S_1)\neq\emptyset$ and $V(C)\cap V(G_2-S_2)\neq\emptyset$.

In this case, let $C_1$ be the subgraph induced by the vertex set $V(C)\cap V(G_1-S_1)$, and let $C_2$ be the subgraph induced by the vertex set $V(C)\cap V(G_2-S_2)$. We have $N_{G_2}(C_1)\subseteq S_2\cup V(C_2)$ and $N_{G_1}(C_2)\subseteq S_1\cup V(C_1)$, then $|V(C_1)|\leq|S_2|+|V(C_2)|$ and $|V(C_2)|\leq|S_1|+|V(C_1)|$. We now consider three subcases.

**Subcase 2.1**: Both $G_1-S_1$ and $G_2-S_2$ are connected.

In this subcase, we have $|S_1|+|V(C_1)|=|V(G_1)|\geq 4(n-2)+2$ and $|S_2|+|V(C_2)|=|V(G_2)|\geq 4(n-2)+2$. Then

$$\begin{aligned}|S|+\left\lceil\frac{|V(C)|}{2}\right\rceil&=|S|+\left\lceil\frac{|V(G)|-|S|}{2}\right\rceil\\&\geq\left\lceil\frac{|V(G)|+|S|}{2}\right\rceil\\&=\left\lceil\frac{|V(G_1)|+|V(G_2)|+|S_1|+|S_2|}{2}\right\rceil\\&\geq 4(n-2)+2.\end{aligned}$$

**Subcase 2.2**: Only one of $G_1-S_1$ and $G_2-S_2$ is connected.

Assume without loss of generality that $G_1-S_1$ is connected and that $G_2-S_2$ is disconnected. Then, $|S_1|+|V(C_1)|=|V(G_1)|\geq 4(n-2)+2$, and $|S_2|\geq n-1\geq 4$. Since $|V(C_1)|\leq|S_2|+|V(C_2)|$, we have

$$\begin{aligned}|S|+\left\lceil\frac{|V(C)|}{2}\right\rceil&=|S_1|+|S_2|+\left\lceil\frac{|V(C_1)|+|V(C_2)|}{2}\right\rceil\\&\geq|S_1|+|S_2|+\left\lceil\frac{|V(C_1)|+(|V(C_1)|-|S_2|)}{2}\right\rceil\\&=(|S_1|+|V(C_1)|)+\left\lceil\frac{|S_2|}{2}\right\rceil\\&\geq 4(n-2)+2.\end{aligned}$$

**Subcase 2.3**: Both $G_1 - S_1$ and $G_2 - S_2$ are disconnected. This subcase is further divided into three cases.

**Subcase 2.3a**: $|S_1| \geq 2(n-2)$ and $|S_2| \geq 2(n-2)$. In this case, $|S| + \lceil |V(C)|/2 \rceil \geq 4(n-2)+2$ for $|V(C)| \geq 4$.

**Subcase 2.3b**: $n-1 \leq |S_1| \leq 2(n-2)-1$ and $n-1 \leq |S_2| \leq 2(n-2)-1$. By Lemma 4, there exists an edge $e=(u,v)$ in $C_1$. Since $e$ has $2(n-2)$ neighbors in $G_1$ and $|S_1| \leq 2(n-2)-1$, then there exists a path $P_3 = \langle u_1, u_2, u_3 \rangle$ of length three in $C_1$ with $\{(u_1,u_2),(u_2,u_3)\} \subseteq E(G_1)$. By Lemma 4, $u_1$ has a neighbor set $X_1$ in $V(G_1 - P_3)$ with order $n-2$; $u_2$ has a neighbor set $X_2$ in $V(G_1-P_3)-X_1$ with order $n-3$; and $u_3$ has a neighbor set $X_3$ in $V(G_1-P_3)-X_1-X_2$ with order at least $n-3$. Then, $|N_{G_1}(P_3)| = |X_1|+|X_2|+|X_3| \geq (n-2)+(n-3)+(n-3) = 3(n-2)-2$. Since $G_1 - S_1$ is disconnected, $N_{G_1}(P_3) \subseteq S_1 \cup V(C_1)$. Thus, we have $|S_1| + |V(C_1)| \geq |N_{G_1}(P_3)| + |P_3| \geq (3(n-2)-2)+3 = 3(n-2)+1$. For the same reason, we get $|S_2| + |V(C_2)| \geq 3(n-2)+1$. Then

$$\begin{aligned} |S| &+ \left\lceil \frac{|V(C)|}{2} \right\rceil \\ &= (|S_1|+|S_2|) + \left\lceil \frac{|V(C_1)|+|V(C_2)|}{2} \right\rceil \\ &\geq (|S_1|+|S_2|) \\ &\quad + \left\lceil \frac{(3(n-2)+1-|S_1|)+(3(n-2)+1-|S_2|)}{2} \right\rceil \\ &\geq 3(n-2)+1+\left\lceil \frac{|S_1|+|S_2|}{2} \right\rceil \\ &\geq 4(n-2)+2. \end{aligned}$$

**Subcase 2.3c**: Either $|S_1| \geq 2(n-2)$, $n-1 \leq |S_2| \leq 2(n-2)-1$, or $|S_2| \geq 2(n-2)$, $n-1 \leq |S_1| \leq 2(n-2)-1$.

Without loss of generality, assume that $|S_2| \geq 2(n-2)$, $n-1 \leq |S_1| \leq 2(n-1)-1$. Then, using the same argument for Subcase 2.3b, we get $|S_1| + |V(C_1)| \geq 3(n-2)+1$. Thus, we have

$$\begin{aligned} |S| + \left\lceil \frac{|V(C)|}{2} \right\rceil &= |S_1|+|S_2| + \left\lceil \frac{|V(C_1)|+|V(C_2)|}{2} \right\rceil \\ &\geq |S_1|+|S_2| \\ &\quad + \left\lceil \frac{(3(n-2)+1-|S_1|)+|V(C_2)|}{2} \right\rceil \\ &= |S_2| + \left\lceil \frac{3(n-2)+1+|S_1|+|V(C_2)|}{2} \right\rceil \\ &\geq 2(n-2) \\ &\quad + \left\lceil \frac{3(n-2)+1+(n-1)+2}{2} \right\rceil \\ &= 4(n-2)+2. \end{aligned}$$

This completes the proof of Lemma 6. ■

Using the foregoing lemmas, we now have the main result.

*Theorem 7:* Suppose $G = (G_1, G_2; M)$ is an MCN, where $G_1$ and $G_2$ are $(n-1)$-regular $(n-1)$-connected triangle-free networks with order no less than $4(n-2)+2$, and $C(G)=2$. Then, $t_c(G) = 4(n-2)+1$ for $n \geq 5$.

*Proof:* Since $C(G)=2$, by Lemma 3, we have $t_c(G) \leq 4(n-2)+1$. By Lemma 6, we get $t_c(G) \geq 4(n-2)+1$. Then, the result holds. ■

## III. Conditional Diagnosability of Hypercubes and Variants

We have considered in Section II the conditional diagnosability of a class of the MCNs. Suppose $G = (G_1, G_2; M)$ is an MCN, where $G_1$ and $G_2$ are $(n-1)$-regular $(n-1)$-connected triangle-free networks with order no less than $4(n-2)+2$ and $C(G)=2$. Then, we have shown in Theorem 7 that $t_c(G) = 4(n-2)+1$.

For the hypercube $Q_n$, the crossed cube $\text{CQ}_n$, the twisted cube $\text{TQ}_n$, and the Möbius cube $\text{MQ}_n$, all these networks are $n$-regular $n$-connected triangle-free MCNs composed of two $(n-1)$-dimensional subcubes. All of these cubes contain the cycle of length four, and every two vertices have at most two common neighbors. Since $V(Q_{n-1}) = V(\text{CQ}_{n-1}) = V(\text{TQ}_{n-1}) = V(\text{MQ}_{n-1}) = 2^{n-1} > 4(n-2)+1$ for $n \geq 5$, then the following results hold.

*Corollary 8 [11]:* The conditional diagnosability of hypercube $Q_n$ is $t_c(Q_n) = 4(n-2)+1$ for $n \geq 5$.

*Corollary 9:* The conditional diagnosability results of the crossed cube $\text{CQ}_n$, the twisted cube $\text{TQ}_n$, and the Möbius cube $\text{MQ}_n$ are $4(n-2)+1$ for $n \geq 5$.

## IV. Conditional Diagnosability of BC Networks

BC networks were introduced in [6], and their conditional diagnosability was studied in [16]. BC networks can recursively be defined as follows: The 1-D BC network $X_1$ is a complete graph with two nodes. Let $L_1 = \{X_1\}$. The $n$-dimensional BC network $X_n$ is defined as follows: $V(X_n) = V(G_1) \cup V(G_2)$ and $E(X_n) = E(G_1) \cup E(G_2) \cup M$, where $G_1 \in L_{n-1}$, $G_2 \in L_{n-1}$, and $M$ is a perfect matching between $V(G_1)$ and $V(G_2)$. $L_n = \{X_n : X_n \text{ is an } n\text{-dimensional BC network}\}$.

There are a few famous networks that belong to BC networks (i.e., hypercube $Q_n$, crossed cube $CQ-n$, and twisted cube $\text{TQ}_n$).

From the definition, we know that the $n$-dimensional BC network $X_n$ is an MCN. Furthermore, we know that $X_n$ is an $n$-regular $n$-connected graph with $2^n$ nodes and $cn(X_n) = 2$ from [6] and [16]. Thus, the $n$-dimensional BC network $X_n$ satisfies the conditions of theorem 7. However, the inverse proposition does not hold. For example, the network $(FQ_{n-1}, FQ_{n-1}; M)$ is a matching network, where $FQ_n$ is an $n$-dimensional folded hypercube that will be defined in the following discussion.

The $n$-dimensional folded hypercube $FQ_n$ is a network with the vertex set $V(FQ_n) = \{x_1, x_2, \ldots, x_n : x_i \in \{0,1\}, i = 1,2,\ldots,n\}$. Two vertices $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ in $FQ_n$ are linked by an edge iff $\sum_{i=1}^{n} |x_i - y_i| = 1$ or $\sum_{i=1}^{n} |x_i - y_i| = n$. The fold hypercube $FQ_{n-1}$ is $n$-regular $n$-connected with $2^{n-1}$ nodes and $cn(FQ_{n-1}) = 2$ for $n \geq 5$ from [15]. The network $(FQ_{n-1}, FQ_{n-1}; M)$ is a matching network satisfying the condition of Theorem 7. However, $(FQ_{n-1}, FQ_{n-1}; M)$ is not a BC network since it is an $(n+1)$-regular graph with $2^n$ vertices. Thus, our conditional diagnosability result applies to a more general class of networks than to BC networks.

## V. Summary

The PMC model introduced by Preparata *et al.* [13] has proved to be a very powerful tool in the fault diagnosis of large-scale systems. Recently, Lai *et al.* [11] have introduced the concept of conditional diagnosability and established the diagnosability of the hypercube under the PMC model. In this brief, we have generalized this result. Specifically, first, we establish the diagnosability of a general class of MCNs under the PMC model. Using this result, we deduce that the conditional diagnosability of the hypercube, the crossed cube, the twisted cube, and the Möbius cube all have the same conditional diagnosability of $4(n-2)+1$, for $n \geq 5$. We show that the BC networks [6], [16] satisfy the conditions of Theorem 7, and thus, our conditional diagnosability result also applies to BC networks. Finally, we show that the MCNs satisfying the conditions of Theorem 7 are more general than the BC networks.

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