On Maximal Planarization of Nonplanar Graphs

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Abstract —**In this paper, we first point out that the planarization algorithm due to Ozawa and Takahashi [4] does not in general produce a maximal planar subgraph when applied on a nonplanar graph. However, we prove that the algorithm produces a maximal planar subgraph in the case of a complete graph.**

I. Introduction

A graph is *planar* if it can be drawn on a plane with no two edges crossing each other except at their end vertices. A subgraph G' of a nonplanar G is a *maximal planar subgraph* of G if G' is planar, and adding to G' any edge of G not present in G' results in a nonplanar subgraph of G. The process of removing a set of edges from a nonplanar graph to obtain a maximal planar subgraph is known as *maximal planarization*. Determining the minimum number of edges whose removal from a nonplanar graph will yield a maximal planar subgraph is an NP-complete problem [1]. However, a few algorithms which attempt to produce maximal planar subgraphs having the largest possible number of edges have been reported.

Recently, Chiba, Nishioka, and Shirakawa [2] modified Hopcroft and Tarjan's planarity testing algorithm [3] to maximally planarize a nonplanar graph. Their algorithm needs $0(mn)$ time and $0(mn)$ space for a nonplanar graph having n vertices and m edges. Ozawa and Takahashi [4] proposed another $0(mn)$ time and $0(m+n)$ space algorithm to planarize a nonplanar graph using the PQ-tree implementation [5] of Lempel, Even, and Cederbaum's planarity testing algorithm [6] (in short, the LEC algorithm).

In this paper, we first discuss in Section II the principle of the planarization approach used by Ozawa and Takahashi [4] and point out that their algorithm does not in general produce a maximal planar subgraph when applied on a nonplanar graph. In Section III, we prove that this algorithm produces a maximal planar subgraph in the case of a complete graph. For terms not defined in this paper, [4]–[6] may be referred to.

II. Principle of an Approach for Planarization

In this section, we discuss the basic principle of an approach for planarization based on the LEC algorithm. Let G denote a simple biconnected st-graph. Let $T_1, T_2, \cdots, T_{n-1}$ be the PQ-trees corresponding to the bush forms of G. For any node X in T_i, we define the frontier of X as the left-to-right order of appearance of the leaves in the subtree of T_i rooted at X. Ozawa and Takahashi [4] classify the nodes of any PQ-tree according to their frontier as follows.

Type W: A node is said to be Type W if its frontier consists of only nonpertinent leaves.

Type B: A node is said to be Type B if its frontier consists of only pertinent leaves.

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Type H: A node X is said to be Type H if the subtree rooted at X can be rearranged such that all the descendant pertinent leaves of X appear consecutively at either the left or the right end of the frontier, with at least one nonpertinent leaf appearing at the other end of the frontier.

Type A: A node X is said to be Type A if the subtree rooted at X can be rearranged such that all the descendant pertinent leaves of X appear consecutively in the middle of the frontier with at least one nonpertinent leaf appearing at each end of the frontier.

The central concept of the planarization algorithm is stated in the following theorem, which is easy to prove.

Theorem 1: An n-vertex graph G is planar if and only if the pertinent roots in all the PQ-trees $T_2, T_3, \cdots, T_{n-1}$ of G are Type B, H, or A.

We call a PQ-tree *reducible* if its pertinent root is Type B, H, or A; otherwise it is *irreducible*. Theorem 1 implies that the graph G is planar if and only if all the T_i's are reducible. If any T_i is irreducible, we can make it reducible by appropriately deleting some of the leaves from it. Of course, we would like to delete a minimum number of leaves while trying to make T_i reducible. If we make all the T_i's reducible this way, then a planar subgraph can be obtained by removing from the nonplanar graph the edges corresponding to the leaves that are deleted.

It is easy to see that the PQ-tree T_{n-1} is always reducible because its root is Type B. The tree T_1 is also reducible because it has only one pertinent leaf—the leaf corresponding to the edge $(1,2)$. Consider now an irreducible PQ-tree T_i of an n-vertex nonplanar graph. For a node X in T_i, let w, b, h, and a be the minimum number of descendant leaves of X which should be deleted from T_i so that X becomes Type W, B, H, and A, respectively. We denote these numbers of a node as $[w,b,h,a]$. Any node in T_i may be made Type W, B, H, or A by appropriately deciding the types of its children. So the $[w,b,h,a]$ number of any node can be computed from that of its children. Thus, to make T_i reducible, we first traverse it bottom-up from the leaves to the pertinent root and compute the $[w,b,h,a]$ number for every node in T_i. Once the $[w,b,h,a]$ number of the pertinent root is computed, we make the pertinent root Type B, H, or A depending on which one of the numbers b, h, and a of the root is the smallest. After determining the type of the pertinent root, we traverse T_i top-down from the pertinent root to the leaves and decide the type of each node in the pertinent subtree T_i. Note that the type of a node uniquely determines the types of its children and so the types of all the leaves in T_i can be determined by this top-down traversal. This information would help us decide the nodes to be deleted from T_i in order to make it reducible. After deleting these nodes from T_i, we can apply the PQ-tree reduction procedure on T_i [5].

Repeating the above procedure for each irreducible T_i, we can obtain a planar subgraph of the nonplanar graph. It is easy to see that if the minimum of b, h, and a for the pertinent root in a PQ-tree T_i is zero, then T_i is reducible. Thus, we can determine whether a T_i is reducible or not from the $[w,b,h,a]$ number of its pertinent root.

Note that the planarization approach described above may not determine a maximal planar subgraph. This can be explained as follows. Suppose we delete certain leaves from T_i to make it reducible. In a later reduction step, some of the leaves which caused the irreducibility of T_i may themselves be deleted. In such a case, we may be able to return to G a subset of the edges which were removed while making T_i reducible. Hence, the planar subgraph obtained may not be maximally planar.

Computing the $[w, b, h, a]$ numbers for the nodes in a PQ-tree is a crucial step in the above approach. Ozawa and Takahashi [4] have presented formulas to compute these numbers. A main drawback of Ozawa and Takahashi's algorithm arises from the fact that they permit deletion of both pertinent and nonpertinent leaves from a tree T_i to make it reducible. Since, in T_i, the pertinent leaves correspond to the edges entering vertex $(i+1)$ in the st-graph G and the nonpertinent leaves correspond to the edges entering vertices greater than $(i+1)$, it may so happen that, as the algorithm proceeds, all the edges entering a vertex $k > (i+1)$ may get removed from G and thus vertex k and some of the other vertices may not be present in the resulting planar subgraph. Thus, the planar subgraph determined by Ozawa and Takahashi's algorithm may not be a spanning subgraph of the given nonplanar graph.

III. On a Property of Ozawa and Takahashi's Algorithm

Ozawa and Takahashi expected their algorithm to produce a maximal planar subgraph in the case of a complete graph. In this section, we prove that this is indeed true.

Theorem 2: In the case of a complete graph, Ozawa and Takahashi's algorithm determines a maximal planar subgraph.

Proof: We prove the theorem by showing that the planar subgraph obtained by Ozawa and Takahashi's algorithm when applied on an n-vertex complete graph will have n vertices and $3n-6$ edges.

Note that, for any graph, the PQ-trees T_2 and T_{n-1} are always reducible, and so no leaves need be deleted from these trees. For any i, $3 \leqslant i \leqslant n-2$, the PQ-tree T_i of an n-vertex complete graph is of the form shown in Fig. 1. The $[w, b, h, a]$ numbers of the nodes in T_i can be computed as follows.

i) For the P-nodes labeled $2, 3, \cdots, i$

$$w = 1$$
$$b = n - i - 1$$
$$h = 0$$
$$a = 0.$$

ii) For the only Q-node

$$w = i - 1$$
$$b = (i-1)(n-i-1)$$
$$h = i - 2$$
$$a = i - 3.$$

iii) For the pertinent root (the P-node labeled 1)

$$w = i$$
$$b = i(n-i-1)$$
$$h = i - 2$$
$$a = i - 2.$$

Thus, from each T_i, $3 \leqslant i \leqslant n-2$, $(i-2)$ leaves are removed to make it reducible. Hence, the total number of edges removed is given by

$$\sum_{i=3}^{n-2} (i-2) = \frac{(n-3)(n-4)}{2}.$$

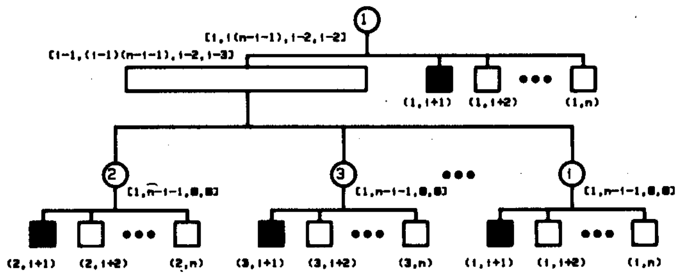

Fig. 1. PQ-tree T_i for an n-vertex complete graph.

Since an n-vertex complete graph has $n(n-1)/2$ edges, the number of edges in the planar graph determined by Ozawa and Takahashi's algorithm is given by

$$\frac{n(n-1)}{2} - \frac{(n-3)(n-4)}{2} = 3n - 6.$$

As can be seen from Fig. 1, for an n-vertex complete graph, minimum leaf deletion in the case of each T_i, $3 \leqslant i \leqslant n-3$, necessarily results in deletion of only pertinent leaves. Since, from each T_i, $3 \leqslant i < n-3$, only $(i-2)$ leaves are removed, it follows that, in each such reducible T_i, there will be exactly two pertinent leaves. On the other hand, in the case of T_{n-2}, minimum leaf deletion can be achieved by deleting either $(n-4)$ pertinent leaves or $(n-4)$ nonpertinent leaves. However, even in this case, irrespective of the choice made, there will be at least two pertinent leaves in the reducible T_{n-2}. Since the edges $(1, n)$ and $(1, 2)$ are not removed, it follows that in the planar subgraph obtained by Ozawa and Takahashi's algorithm, each vertex will be connected to at least one lower numbered vertex, and so this subgraph will be connected and will have n vertices. Hence, the theorem. □

IV. Conclusions

In this paper, we have pointed out that Ozawa and Takahashi's planarization algorithm does not in general produce a maximal planar subgraph when applied on a nonplanar graph. We have proved that this algorithm produces a maximal planar subgraph in the case of a complete graph.

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