Inverse of a Non-Singular Submatrix of a Reduced Incidence Matrix

Let A_{12} be a non-singular submatrix of a reduced incidence matrix of a connected graph. The non-zero entries of any row of A_{12}^{-1} are either all +1 or all −1.

The proof now follows: consider a cut-set matrix Q of a connected graph G having v vertices. Let A be the reduced incidence matrix of G with vertex r as reference. Let row i of Q correspond to cut-set q_i of G. Let the vertices of G be partitioned into two sets C_i and B_i when the edges of q_i are removed from G. Then it is well known that q_i contains only those edges which have one vertex in C_i and the other vertex in B_i.

Let C_i contain the reference vertex r. Let A_j refer to the row of A corresponding to vertex j of G. If Q_i refers to the ith row of Q then

$$Q_i = \begin{cases} +\sum_{j \in B_i} A_j\,, & \text{if orientation of cut-set } q_i \text{ is away from } B_i\,. \\ -\sum_{j \in B_i} A_j\,, & \text{if orientation of cut-set } q_i \text{ is toward } B_i\,. \end{cases} \tag{1}$$

The above relationship is a consequence of the fact that the sum of any two rows A_j and A_k of A consists of non-zero entries only in those columns which correspond to the edges which are incident either at vertex j or at vertex k but not at both. It should be pointed out that in forming A the definition given in Ref. 1 is followed.

It follows from the above discussion that each row of Q can be expressed as a linear combination of the rows of A, the non-zero coefficients of the linear combination being either all 1 or all −1. Hence we can conclude that Q can be written as

$$Q = DA$$

where the matrix D has the property that the non-zero entries of any of its rows are either all 1 or all −1.

Consider next any non-singular $(v-1)\times(v-1)$ submatrix A_{12} of A. Let the tree, the edges of which correspond to the columns of A_{12}, be denoted by T. Let Q_f be the fundamental cut-set matrix of G with respect to T. Then Q_f can be written as

$$Q_f = DA\,.$$

But it is known that

$$D = A_{12}^{-1} \tag{2}$$

Hence it follows from (1) and (2) that the non-zero entries in any row of A_{12}^{-1} are either all 1 or all −1 when A_{12} is of order $(v-1)$. Further any non-singular $k \times k$ submatrix of A is a submatrix of a reduced incidence matrix of a connected graph containing $(k+1)$ vertices. Hence we conclude that the non-zero entries in any row of the inverse of any non-singular submatrix of A are either all 1 or all −1. Hence the theorem.

A procedure for determining the inverse now follows: consider any non-singular $(m \times m)$ matrix A_{12} which consists of at most a 1 or a −1 per column. Let all other entries of A_{12} be zero. We now establish a procedure for the determination of the inverse of A_{12} *without evaluating cofactors*. This problem has been considered earlier[2,3,4].

The graph T which has A_{12} as a reduced incidence matrix can be drawn by inspection of A_{12}. Let r be the reference vertex, with respect to which A_{12} is the reduced incidence matrix of T. It should be noted that T is a tree. Let G be any connected graph constructed on the vertices of T such that T is a subgraph of G. Let Q_f be the fundamental cut-set matrix of G with respect to T. If A is the reduced incidence matrix of G with vertex r as reference then A_{12} is a submatrix of A. If the columns of A and Q_f are arranged in the same order then Q_f can be written as

$$Q_f = D A$$

But it is well known that $D = A_{12}^{-1}$. Further the entries of D can be determined using (1). Let T consist of m edges $e_1, e_2, \ldots, e_m$. Let the row i of A correspond to vertex i. Let column i of A correspond to edge e_i of T. Let row i of Q_f correspond to edge e_i of T. Let the vertices of T be separated into two sets C_i and B_i when e_i is removed. Let the reference vertex r be in C_i. Then using (1) we can obtain the (i, j) entry d_{ij} of D as follows.

$$d_{ij} = \begin{cases} 0, & \text{if vertex } j \text{ is in } C_i \\ 1, & \text{if vertex } j \text{ is in } B_i \text{ and edge } e_i \text{ is oriented away from } B_i \\ -1, & \text{if vertex } j \text{ is } e_i \text{ and edge } e_i \text{ is oriented toward } B_i \end{cases}$$

An example is now given: let it be required to determine the inverse of the matrix A_{12} given below.

$$A_{12} = \text{vertices of } T \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \overset{\text{edges of } T}{\begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array}} $$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The tree T having A_{12} as its reduced incidence matrix with vertex 7 as reference is shown in Fig. 1. Consider the vertex sets C_3 and B_3 into which the vertices of T are separated when e_3 is removed from T.

$$C_3 = \{1, 7, 2, 3\}$$
$$B_3 = \{4, 5, 6\}$$

Since vertices 4, 5 and 6 are in B_3 and e_3 is oriented away from B_3

$$d_{14} = d_{35} = d_{36} = 1$$

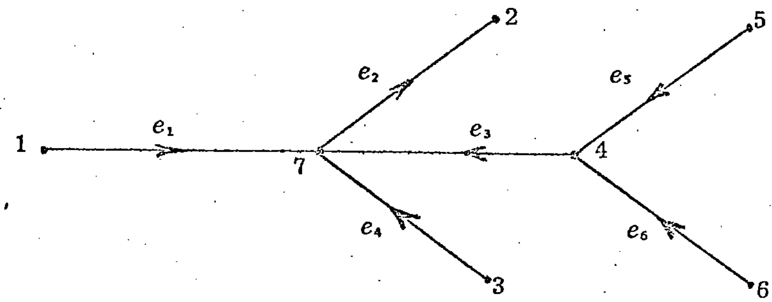

Fig. 1.

Also, since vertices 1, 2 and 3 are in C_3 which contains the reference vertex 7

$$d_{31} = d_{32} = d_{33} = 0$$

Similarly other entries of the matrix $D = A_{12}^{-1}$ can be determined.

$$A_{12}^{-1} = \begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{array} \begin{array}{c} \begin{matrix} 1 & \ \ 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

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