8

Signal Flow Graphs

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8.1 Introduction

Signal flow graph theory is concerned with the development of a graph theoretic approach to solving a system of linear algebraic equations. Two closely related methods proposed by Coates [1] and Mason [2, 3] have appeared in the literature and have served as elegant aids in gaining insight into the structure and nature of solutions of systems of equations. In this chapter we develop these two methods. Our development follows closely [4].

An extensive discussion of signal flow theory may be found in [5]. Applications of signal flow theory in the analysis and synthesis electrical networks may be found in Sections 4 and 5. Coates' and Mason's methods may be viewed as generalizations of a basic theorem in graph theory due to Harary [6], which provides a formula for finding the determinant of the adjacency matrix of a directed graph. Thus, our discussion begins with the development of this theorem. For graph theoretic terminology the reader may refer to Chapter 7.

8.2 Adjacency Matrix of a Directed Graph

Consider a directed graph $G = (V, E)$ with no parallel edges. Let $V = \{v_1, \ldots, v_n\}$. The **adjacency matrix** $M = [m_{ij}]$ of G is an $n \times n$ matrix defined as follows:

$$m_{ij} = \begin{cases} 1, & \text{if } (v_i, v_j) \in E \\ 0, & \text{otherwise} \end{cases}$$

The graph shown in Fig. 8.1 has the following adjacency matrix:

$$M = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \left[\begin{array}{cccc} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{array}\right] \end{array}$$

In the following we shall develop a topological formula for det M. Toward this end we introduce some basic terminology. A **1-factor** of a directed graph G is a spanning subgraph of G in which the

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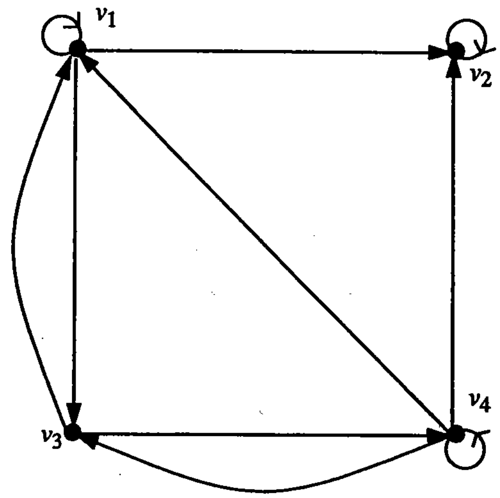


8.1 The graph G.

in-degree and the out-degree of every vertex are both equal to 1. It is easy to see that a 1-factor is a collection of vertex-disjoint directed circuits. Because a self-loop at a vertex contributes 1 to the in-degree and 1 to the out-degree of the vertex, a 1-factor may have some self-loops. As an example, the three 1-factors of the graph of Fig. 8.1 are shown in Fig. 8.2.

A **permutation** $(j_1, j_2, \ldots, j_n)$ of integers $1, 2, \ldots, n$ **is even (odd)** if an even (odd) number of interchanges are required to rearrange it as $(1, 2, \ldots, n)$. The notation

$$\begin{pmatrix} 1, 2, \ldots, n \\ j_1, j_2, \ldots, j_n \end{pmatrix}$$

is also used to represent the permutation $(j_1, j_2, \ldots, j_n)$. As an example, the permutation $(4, 3, 1, 2)$ is odd because it can be rearranged as $(1, 2, 3, 4)$ using the following sequence of interchanges:

1. Interchange 2 and 4.
2. Interchange 1 and 2.
3. Interchange 2 and 3.

For a permutation $(j) = (j_1, j_2, \ldots, j_n)$, $\varepsilon_{j_1, j_2, \ldots, j_n}$, is defined as equal to 1, if (j) is an even permutation; otherwise, $\varepsilon_{j_1, j_2, \ldots, j_n}$, is equal to -1.

Given an $n \times n$ square matrix $X = [x_{ij}]$, we note that $\det X$ is given by

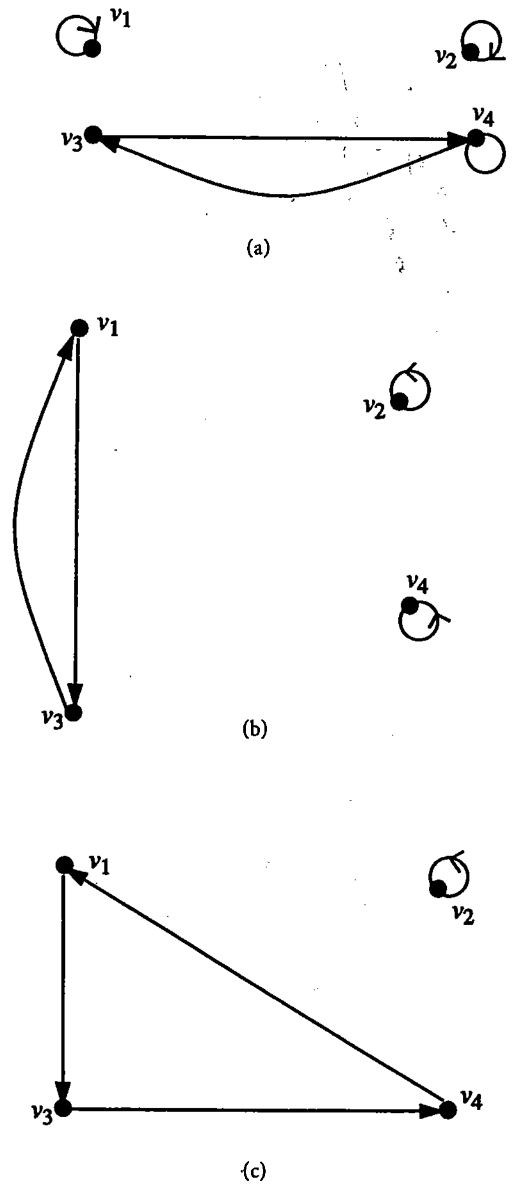
$$\det X = \sum_{(j)} \varepsilon_{j_1, j_2, j_3, \ldots, j_n} \quad x_{1j_1}, x_{2j_2} \cdots x_{nj_n}$$

where the summation $\sum_{(j)}$ is over all permutations of $1, 2, \ldots, n$ [7].

The following theorem is due to Harary [6].

THEOREM 8.1 *Let H_i, $i = 1, 2, \ldots, p$ be the 1-factors of an n-vertex directed graph G. Let L_i denote the number of directed circuits in H_i, and let M denote the adjacency matrix of G. Then*

$$\det M = (-1)^n \sum_{i=1}^{p} (-1)^{L_i}$$



8.2 The three 1-factors of the graph of Fig. 8.1.

PROOF 8.1 From the definition of a determinant, we have

$$\det M = \sum_{(j)} \varepsilon_{j_1, j_2, \ldots, j_n} \quad m_{1j_1} \cdot m_{2j_2} \cdots m_{nj_n} \tag{8.1}$$

Proof will follow if we establish the following:

1. Each nonzero term $m_{1j_1} \cdot m_{2j_2} \cdots m_{nj_n}$ corresponds to a 1-factor of G, and conversely, each 1-factor of G corresponds to a non-zero term $m_{1j_1} \cdot m_{2j_2} \cdots m_{nj_n}$.
2. $\varepsilon_{j_1, j_2, \ldots, j_n} = (-1)^{n+L}$ if the 1-factor corresponding to a nonzero $m_{1j_1} \cdot m_{2j_2} \cdots m_{nj_n}$ has L directed circuits.

A nonzero term $m_{1j_1}m_{1j_2}\cdots m_{nj_n}$ corresponds to the set of edges $(v_1, v_{j1}), (v_2, v_{j2})\cdots(v_n, v_{jn})$. Each vertex appears exactly twice in this set, once as an initial vertex and once as a terminal vertex of a pair of edges. Therefore, in the subgraph induced by these edges, for each vertex its in-degree and its out-degree are both equal to 1, and this subgraph is a 1-factor of G. In other words, each non-zero term in the sum in (8.1) corresponds to a 1-factor of G. The fact that each 1-factor of G corresponds to a non-zero term $m_{1j_1}\cdot m_{2j_2}\cdots m_{nj_n}$ is obvious.

As regards $\varepsilon_{j_1,j_2,\ldots,j_n}$, consider a directed circuit C in the 1-factor corresponding to $m_{1j_1}\cdot m_{2j_2}\cdots m_{nj_n}$. Without loss of generality, assume that C consists of the w edges

$$(v_1, v_2),\quad (v_2, v_3),\quad \cdots,\quad (v_w, v_1)$$

It is easy to see that the corresponding permutation $(2, 3, \ldots, w, 1)$ can be rearranged as $(1, 2, \ldots, w)$ using $w-1$ interchanges. If the 1-factor has L directed circuits with lengths $w_1, \ldots, w_L$, the permutation $(j_1, \ldots, j_n)$ can be rearranged as $(1, 2, \ldots, n)$ using

$$(w_1-1)+(w_2-1)+\cdots+(w_L-1)=n-L$$

interchanges. So,

$$\varepsilon_{j_1,j_2,j_n}=(-1)^{n+L}$$

□

As an example, for the 1-factors (shown in Fig. 8.2) of the graph of Fig. 8.1, the corresponding L_i are $L_1 = 3$, $L_2 = 3$, and $L_3 = 2$. So, the determinant of the adjacency matrix of the graph of Fig. 8.1 is

$$(-1)^4\left[(-1)^3+(-1)^3+(-1)^2\right]=-1$$

Consider next a weighted directed graph G in which each edge (v_i, v_j) as associated with a weight w_{ij}. Then we may define the **adjacency matrix** $M = [m_{ij}]$ of G as follows:

$$m_{ij}=\begin{cases} w_{ij} & \text{if } (v_i, v_j)\in E \\ 0, & \text{otherwise}\end{cases}$$

Given a subgraph H of G, let us define weight $w(H)$ of H as the product of the weights of all edges in H. If H has no edges, then we define $w(H) = 1$. The following result is an easy generalization of Theorem 8.1.

THEOREM 8.2 *The determinant of the adjacency matrix of an n-vertex directed graph G is given by*

$$\det M=(-1)^n\sum_H(-1)^{L_H}w(H),$$

where H is a 1-factor, $w(H)$ is the weight of H, and L_H is the number of directed circuitsin H.

8.3 Coates' Gain Formula

Consider a linear system described by the equation

$$AX = Bx_{n+1} \tag{8.2}$$

where A is a nonsingular $n \times n$ matrix, X is a column vector of unknown variables $x_1, x_2, \ldots, x_n$, B is a column vector of elements $b_1, b_2, \ldots, b_n$ and x_{n+1} is the input variable. It is well known that

$$\frac{x_k}{x_{n+1}} = \frac{\sum_{i=1}^{n} b_i \Delta_{ik}}{\det A} \tag{8.3}$$

where Δ_{ik} is the (i, k) cofactor of A.

To develop Coates' topological formulas for the numerator and the denominator of (8.3), let us first augment the matrix A by adding $-B$ to the right of A and adding a row of zeroes at the bottom of the resulting matrix. Let this matrix be denoted by A'. **The Coates flow graph**[1] $G_c(A')$, or simply the **Coates graph** associated with matrix A', is a weighted directed graph whose adjacency matrix is the transpose of the matrix A'. Thus, $G_c(A')$ has $n+1$ vertices $x_1, x_2, \ldots, x_{n+1}$, and if $a_{ji} \cdots \neq 0$, then $G_c(A')$ has an edge directed from x_i to x_j with weight a_{ji}. Clearly, the Coates graph $G_c(A)$ associated with matrix A can be obtained from $G_c(A')$ by removing the vertex x_{n+1}.

As an example, for the following system of equations

$$\begin{bmatrix} 3 & -2 & 1 \\ -1 & 2 & 0 \\ 3 & -2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} x_4 \tag{8.4}$$

the matrix A' is

$$A' = \begin{bmatrix} 3 & -2 & 1 & -3 \\ -1 & 2 & 0 & -1 \\ 3 & -2 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The Coates graphs $G_c(A')$ and $G_c(A)$ are shown in Fig. 8.3.

Because a matrix and its transpose have the same determinant value and because A is the transpose of the adjacency matrix of $G_c(A')$, we obtain the following result from Theorem 8.2.

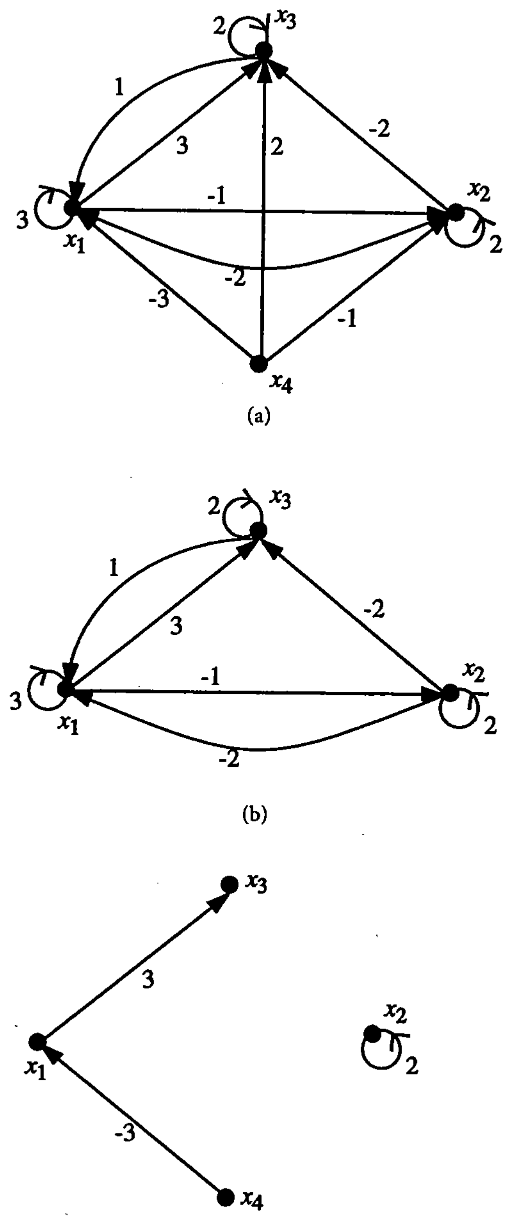
THEOREM 8.3 *If a matrix A is nonsingular, then*

$$\det A = (-1)^n \sum_H (-1)^{L_H} w(H) \tag{8.5}$$

where H is a 1-factor of $G_c(A)$, $w(H)$ is the weight of H and L_H is the number of directed circuits in H.

To derive a similar expression for the sum in the numerator of (8.3), we first define the concept of a **1-factorial connection.** A 1-factorial connection H_{ij} from x_i to x_j in $G_c(A)$ is a spanning subgraph of G which contains a directed path P from x_i to x_j and a set of vertex-disjoint directed circuits which include all the vertices of $G_c(A)$ other than those which lie on P. Similarly, a 1-factorial connection of $G_c(A')$ can be defined. As an example, a 1-factorial connection from x_4 to x_3 of the graph $G_c(A')$ of Fig. 8.3(a) is shown in Fig. 8.3(c).

[1] In network and systems theory literature, the Coates graph is referred to as a **flow graph.**



8.3 (a) The Coates graph $G_c(A')$; (b) the graph $G_c(A)$; (c) A-factorial connection $H_{4,3}$ of the graph $G_c(A')$

THEOREM 8.4 *Let $G_c(A)$ be the Coates graph associated with an $n \times n$ matrix A. Then*

$$1. \quad \Delta_{ii} = (-1)^{n-1} \sum_{H} (-1)^{L_H} w(H)$$

$$2. \quad \Delta_{ij} = (-1)^{n-1} \sum_{H_{ij}} (-1)^{L'_H} w(H_{ij}) \qquad i \neq j$$

where H is a 1-factor in the graph obtained by removing vertex x_i from $G_c(A)$, H_{ij} is a 1-factorial connection in $G_c(A)$ from vertex x_i to vertex x_j, and L_H and L'_H are the numbers of directed circuits in H and H_{ij}, respectively.

PROOF 8.2 1. Note that Δ_{ii} is the determinant of the matrix obtained from A by removing its row i and column i. Also, the Coates graph of the resulting matrix can be obtained from $G_c(A)$ by removing vertex x_i. Proof follows from these observations and Theorem 8.3.

2. Let A_α denote the matrix obtained from A by replacing its jth column by a column of zeroes, except for the element in row i, which is 1. Then it is easy to see that

$$\Delta_{ij} = \det A_\alpha$$

Now, the Coates graph $G_c(A_\alpha)$ can be obtained from $G_c(A)$ by removing all edges incident out of vertex x_j and adding an edge directed from x_j to x_i with weight 1. Then from Theorem 8.3, we get

$$\begin{aligned} \Delta_{ij} &= \det A_\alpha \\ &= (-1)^n \sum_{H_\alpha} (-1)^{L_\alpha} w(H_\alpha) \end{aligned} \tag{8.6}$$

where H_α is a 1-factor of $G_c(A_\alpha)$ and L_α is the number of directed circuits in H_α.

Consider now a 1-factor H_α in $G_c(A_\alpha)$. Let C be the directed circuit of H_α containing x_i. Because in $G_c(A_\alpha)$, (x_j, x_i) is the only edge incident out of x_j, it follows that x_j also lies in C. If we remove the edge (x_j, x_i) from H_α we get a 1-factorial connection, H_{ij}. Furthermore, $L'_H = L_\alpha - 1$ and $w(H_{ij}) = w(H_\alpha)$ because (x_j, x_i) has weight equal to 1. Thus, each H_α corresponds to a 1-factorial connection H_{ij} of $G_c(A_\alpha)$ with $w(H_\alpha) = w(H_{ij})$ and $L'_H = L_\alpha - 1$. The converse of this is also easy to see. Thus, in (8.6) we can replace H_α by H_{ij} and L_α by $(L'_H + 1)$. Then we obtain

$$\Delta_{ij} = (-1)^{n-1} \sum_{H_{ij}} (-1)^{L'_H} w(H_{ij})$$

□

Having shown that each Δ_{ij} can be expressed in terms of the weights of the 1-factorial connections H_{ij} in $G_c(A)$, we now show that $\sum b_i \Delta_{ik}$ can be expressed in terms of the weights of the 1-factorial connections $H_{n+1,k}$ in $G_c(A')$.

First, note that adding the edge (x_{n+1}, x_i) to H_{ik} results in a 1-factorial connection $H_{n+1,k}$, with $w(H_{n+1,k}) = -b_i\, w(H_{ik})$. Also, $H_{n+1,k}$ has the same number of directed circuits as H_{ik}. Conversely, from each $H_{n+1,k}$ that contains the edge (x_{n+1}, x_i) we can construct a 1-factorial connection H_{ik} satisfying $w(H_{n+1,k}) = -b_i\, w(H_{ik})$. Also, $H_{n+1,k}$ and the corresponding H_{ik} will have the same number of directed circuits. Thus, a one-to-one correspondence exists between the set of all 1-factorial connections $H_{n+1,k}$ in $G_c(A')$ and the set of all 1-factorial connections in $G_c(A)$ of the form H_{ik} such that each $H_{n+1,k}$ and the corresponding H_{ik} have the same number of directed circuits and satisfy the relation $w(H_{n+1,k}) = -b_i\, w(H_{ik})$. Combining this result with Theorem 8.4, we get

$$\sum_{i=1}^{n} b_i \Delta_{ik} = (-1)^n \sum_{H_{n+1,k}} (-1)^{L'_H} w(H_{n+1,k}) \tag{8.7}$$

where the summation is over all 1-factorial connections, $H_{n+1,k}$ in $G_c(A')$, and L'_H is the number of directed circuits in $H_{n+1,k}$. From (8.5) and (8.7) we get the following theorem.

THEOREM 8.5 *If the coefficient matrix A is nonsingular, then the solution of (8.2) is given by*

$$\frac{x_k}{x_{n+1}} = \frac{\sum_{H_{n+1,k}} (-1)^{L'_H} w(H_{n+1,k})}{\sum_H (-1)^{L_H} w(H)} \tag{8.8}$$

for $k = 1, 2, \ldots, n$, where $H_{n+1,k}$ is a 1-factorial connection of $G_c(A')$ from vertex x_{n+1} to vertex x_k, H is a 1-factor of $G_c(A)$, and L'_H and L_H are the numbers of directed circuits in $H_{n+1,k}$ and H, respectively.

Equation (8.8) is the called **Coates' gain formula**. We now illustrate Coates' method by solving the system (8.4) for x_2/x_4. First, we determine the 1-factors of the Coates' graph $G_c(A)$ shown in Fig. 8.3(b). These 1-factors, along with their weights, are listed below. The vertices enclosed within parentheses represent a directed circuit.

1-Factor H	Weight $w(H)$	L_H
$(x_1)(x_2)(x_3)$	12	3
$(x_2)(x_1, x_3)$	6	2
$(x_3)(x_1, x_2)$	4	2
(x_1, x_2, x_3)	2	1

From the above we get the denominator in (8.8) as

$$\sum_H (-1)^{L_H} w(H) = (-1)^3 \cdot 12 + (-1)^2 \cdot 6 + (-1)^2 \cdot 4 + (-1)^1 \cdot 2 = -4$$

To compute the numerator in (8.8) we need to determine the 1-factorial connections $H_{4,2}$ in the Coates graph $G_c(A')$ shown in Fig. 8.3(a). They are listed below along with their weights. The vertices in a directed path from x_4 to x_2 are given within parentheses.

1-Factorial connection $H_{4,2}$	$w(H_{4,2})$	L'_H
$(x_4, x_1, x_2)(x_3)$	6	1
$(x_4, x_2)(x_1)(x_3)$	−6	2
$(x_4, x_2)(x_1, x_3)$	−3	1
(x_4, x_3, x_1, x_2)	−2	0

From the above we get the numerator in (8.8) as

$$\sum_{H_{4,2}} (-1)^{L'_H} w(H_{4,2}) = (-1)^1 \cdot 6 + (-1)^2(-6) + (-1)^1(-3) + (-1)^0(-2) = -11$$

Thus, we get

$$\frac{x_2}{x_4} = \frac{11}{4}$$

8.4 Mason's Gain Formula

Consider again the system of equations

$$AX = Bx_{n+1}$$

We can rewrite the above as

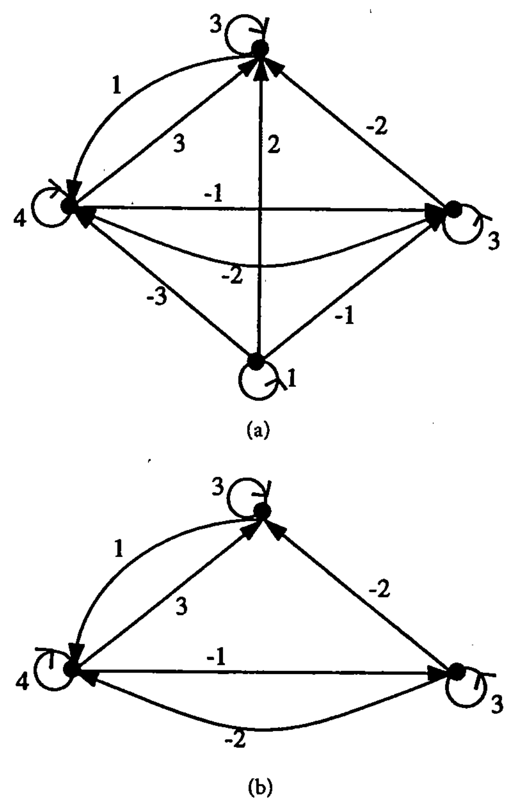
$$x_j = (a_{jj} + 1)x_j + \sum_{\substack{k=1 \\ k \neq j}}^{n} a_{jk}x_k - b_j x_{n+1}, \quad j = 1, 2, \ldots, n, \qquad x_{n+1} = x_{n+1} \tag{8.9}$$

Letting X' denote the column vector of the variables $x_1, x_2, \ldots x_{n+1}$, and U_{n+1} denote the unit matrix of order n, we can write (8.9) in matrix form as follows:

$$(A' + U_{n+1})X' = X' \tag{8.10}$$

where A' is the matrix defined earlier in Section 8.3.

The Coates graph $G_c(A' + U_{n+1})$ is called the **Mason's signal flow graph** or simply the **Mason graph**[2] associated with A', and it is denoted by $G_m(A')$. The Mason graph $G_m(A)$ is defined in a similar manner. The Mason graphs $G_m(A')$ and $G_m(A)$ associated with the system (8.4) are shown in Fig. 8.4. Mason's graph elegantly represents the flow of variables in a system. If we associate each vertex with a variable and if an edge is directed from x_i to x_j, then we may consider the variable x_i as contributing $(a_{ji}x_i)$ to the variable x_j. Thus, x_j is equal to the sum of the products of the weights of the edges incident into vertex x_j and the variables corresponding to the vertices from which these edges emanate.



8.4 (a) The Mason graph $G_m(A')$; (b) the Mason graph $G_m(A)$.

Note that to obtain the Coates graph $G_c(A)$ from the Mason graph $G_m(A)$ we simply subtract one from the weight of each self-loop. Equivalently, we may add at each vertex of the Mason graph a self-loop of weight -1. Let S denote the set of all such loops of weight -1 added to construct the Coates graph G_c from the Mason graph $G_m(A)$.

[2] In network and systems theory literature Mason graphs are usually referred to as **signal flow graphs**.

Consider now the Coates graph G_c constructed as above and a 1-factor H in G_c having j self-loops from the set S. If H has a total of $L_Q + j$ directed circuits, then removing the j self-loops from H will result in a subgraph Q of $G_m(A)$ which is a collection of L_Q vertex disjoint directed circuits. Also,

$$w(H) = (-1)^j w(Q)$$

Then, from Theorem 8.3 we get

$$\begin{aligned} \det A &= (-1)^n \sum_H (-1)^{L_Q + j} w(H) \\ &= (-1)^n \sum_Q (-1)^{L_Q} w(Q) \\ &= (-1)^n \left[1 + \sum_Q (-1)^{L_Q} w(Q)\right] \end{aligned} \tag{8.11}$$

We can rewrite the above as:

$$\det A = (-1)^n \left[1 - \sum_j Q_{j1} + \sum_j Q_{j2} - \sum_j Q_{j3} \cdots \right] \tag{8.12}$$

where each term in $\sum_j Q_{ji}$ is the weight of a collection of i vertex-disjoint directed circuits in $G_m(A)$.

Suppose we refer to $(-1)^n \det A$ as the determinant of the graph $G_m(A)$. Then, starting from $H_{n+1,k}$ and reasoning exactly as above we can express the numerator of (8.3) as

$$\sum_{i=1}^{n} b_i \Delta_{ik} = (-1)^n \sum_j w(P^j_{n+1,k}) \Delta_j \tag{8.13}$$

where $P^j_{n+1,k}$ is a directed path from x_{n+1} to x_k of $G_m(A')$ and Δ_j is the determinant of the subgraph of $G_m(A')$ which is vertex-disjoint from the path $P^j_{n+1,k}$. From (8.12) and (8.13) we get the following theorem.

THEOREM 8.6 *If the coefficient matrix A is (8.2) is nonsingular, then*

$$\frac{x_k}{x_{n+1}} = \frac{\sum_j w(P^j_{n+1,k}) \Delta_j}{\Delta}, \qquad k = 1, 2, \ldots, n \tag{8.14}$$

where $P^j_{n+1,k}$ is the jth directed path from x_{n+1} to x_k of $G_m(A')$, Δ_j is the determinant of the subgraph of $G_m(A')$ which is vertex-disjoint from the jth directed path $P^j_{n+1,k}$, and Δ is the determinant of the graph $G_m(A)$.

Equation (8.14) is known as **Mason's gain formula**. In network and systems theory $P^j_{n+1,k}$ is referred to as a **forward path** from vertex x_{n+1} to vertex x_k. The directed circuits of $G_m(A')$ are called the **feedback loops**.

We now illustrate Mason's method by solving the system (8.4) for x_2/x_4. To compute the denominator in (8.14) we determine the different collections of vertex-disjoint directed circuits of the Mason graph $G_m(A)$ shown in Fig. 8.4(b). They are listed below along with their weights.

Collection of Vertex-Disjoint Directed Circuits of $G_m(A)$	Weight	No. of Directed Circuits
(x_1)	04	1
(x_2)	03	1
(x_3)	03	1
(x_1, x_2)	02	1
(x_1, x_3)	03	1
(x_1, x_2, x_3)	02	1
$(x_1)(x_2)$	12	2
$(x_1)(x_3)$	12	2
$(x_2)(x_3)$	09	2
$(x_2)(x_1, x_3)$	09	2
$(x_3)(x_1, x_2)$	06	2
$(x_1)(x_2)(x_3)$	36	3

From the above we obtain the denominator in (8.14)

$$\begin{aligned}\Delta &= 1 + (-1)^1[4 + 3 + 3 + 2 + 3 + 2] \\ &\quad +(-1)^2[12 + 12 + 9 + 9 + 6] + (-1)^3\, 36 = -4\end{aligned}$$

To compute the numerator in (8.14) we need the forward paths in $G_m(A')$ from x_4 to x_2. They are listed below with their weights.

j	$P^j_{4,2}$	Weight
1	(x_4, x_2)	-1
2	(x_4, x_1, x_2)	3
3	(x_4, x_3, x_1, x_2)	-2

The directed circuits which are vertex-disjoint from $P^1_{4,2}$ are (x_1), (x_3), (x_1, x_3). Thus

$$\Delta_1 = 1 - (4 + 3 + 3) + 12 = 1 - 10 + 12 = 3.$$

(x_3) is the only directed circuit which is vertex-disjoint from $P^2_{4,2}$. So,

$$\Delta_2 = 1 - 3 = -2.$$

No directed circuit is vertex-disjoint from $P^3_{4,2}$, so $\Delta_3 = 1$. Thus, the numerator in (8.14) is

$$P^1_{4,2}\Delta_1 + P^1_{4,2}\Delta_2 + P^1_{4,3}\Delta_3 = -3 - 6 - 2 = -11$$

and

$$\frac{x_2}{x_4} = \frac{11}{4}$$

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