Similarity of graphs and enumeration of dissimilar n^{th} order symmetric sign patterns

Similarité de graphiques et énumération de diagrammes de signes symétriques de $n^{ième}$ ordre dissemblables

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This paper introduces the concepts of similarity of n^{th} order symmetric sign patterns and similarity of n-vertex labelled directed graphs. Two sign patterns are similar if and only if their corresponding directed graphs are similar. The main contributions of this paper are a formula to count the maximum number of mutually dissimilar directed graphs and two algorithms. One algorithm generates a set of mutually dissimilar directed graphs and the other tests similarity of two directed graphs.

Cette étude présente les concepts de similarité des diagrammes de signes symétriques de $n^{ième}$ ordre et de similarité de graphiques orientés marqués à n-sommets. Deux diagrammes de signes sont similaires si, et seulement si, leurs graphes orientés correspondants sont similaires. Une formule est présentée pour compter le nombre maximum de graphes orientés mutuellement dissemblables et deux algorithmes—un pour créer un ensemble de graphes orientés mutuellement dissemblables et l'autre pour tester la similarité de deux graphes orientés.

Introduction

Studies in electrical network theory have given rise to several mathematical concepts. This paper is concerned with a generalization of a graph theoretic question which arose while studying the problem of synthesizing resistance n-port networks.

Consider an $(n \times n)$ real symmetric matrix $Y = [y_{ij}]$. The sign pattern $Y^{(s)} = [y_{ij}^{(s)}]$ of Y is an $(n \times n)$ symmetric matrix difined as follows:

$$y_{ij}^{(s)} = \begin{cases} +, \text{ if } y_{ij} \text{ is positive} \\ -, \text{ if } y_{ij} \text{ is negative} \\ 0, \text{ if } y_{ij} = 0. \end{cases}$$

For example, for the matrix

$$Y = \begin{bmatrix} 3 & 2 & -4 & -1 \\ 2 & 4 & 2 & -4 \\ -3 & 2 & -5 & 0 \\ -1 & -4 & 0 & 6 \end{bmatrix}$$

the sign pattern $Y^{(s)}$ is given by

$$Y^{(s)} = \begin{bmatrix} + & + & - & - \\ + & + & + & - \\ - & + & - & 0 \\ - & - & 0 & + \end{bmatrix}$$

Similarity of symmetric sign patterns

A symmetric sign pattern $Y^{(s)}$ is said to be of strength m if the number of non-zero entries above the diagonal of $Y^{(s)}$ is equal to m. Two sign patterns $Y_1^{(s)}$ and $Y_2^{(s)}$ of equal strength m are similar if $Y_1^{(s)}$ can be made identical to $Y_2^{(s)}$ by changing the signs of all the entries in some rows and the corresponding columns of $Y_1^{(s)}$. Otherwise, $Y_1^{(s)}$ and $Y_2^{(s)}$ are dissimilar.

For example, the sign patterns $Y_1^{(s)}$ and $Y_2^{(s)}$ given below are similar since $Y_2^{(s)}$ can be obtained from $Y_1^{(s)}$ after changing the signs of all the entries in the first and third rows and the corresponding columns of $Y_1^{(s)}$.

$$Y_1^{(s)} = \begin{bmatrix} + & - & + & - \\ - & + & + & - \\ + & + & + & + \\ - & - & + & + \end{bmatrix} ; Y_2^{(s)} = \begin{bmatrix} + & + & + & + \\ + & + & - & - \\ + & - & + & - \\ + & - & - & + \end{bmatrix}$$

Note that the concept of similarity is defined here only with respect to sign patterns of equal strength. Further, if $Y_1^{(s)}$ and $Y_2^{(s)}$ are similar, then each diagonal entry of $Y_1^{(s)}$ is equal to the corresponding diagonal entry of $Y_2^{(s)}$. For this reason, all the diagonal entries are assumed to be zero in the sign patterns considered in this paper. This assumption will involve no loss of generality as far as the questions discussed in this paper are concerned.

Sign patterns occur in the problem of realizability of cutset matrices of directed graphs. They also occur in the problem of realization of multi-port networks based on tree-port structures.[1] The need for introducing the similarity concept arose while studying procedures for realizing third-order real symmetric matrices as the short-circuit conductance or open-circuit resistance matrices of resistance networks. In the case of (3×3) matrices, there are eight different symmetric sign patterns and they are shown below.

$$\text{Group A}\begin{cases} \begin{matrix} y_{12}^{(s)} & y_{13}^{(s)} & y_{23}^{(s)} \\ - & - & - \\ - & + & + \\ + & + & - \\ + & - & + \end{matrix} \end{cases} ;\ \text{Group B}\begin{cases} \begin{matrix} y_{12}^{(s)} & y_{13}^{(s)} & y_{23}^{(s)} \\ + & + & + \\ + & - & - \\ - & - & + \\ - & + & - \end{matrix} \end{cases}$$

It can be shown that sign patterns in group A (B) are similar, and no sign pattern in group A is similar to any sign pattern in group B.[2] Thus, while developing procedures for the synthesis of third-order real symmetric matrices, the discussion should be confined to two distinct sign patterns—one pattern from group A and the other from group B.[2]

In this paper, we count the total number of mutually dissimilar n^{th} order symmetric patterns, and also develop algorithms to generate a maximal set of mutually dissimilar n^{th} order patterns and to test similarity of two sign patterns. For graph-theoretic definitions and notations, we follow methods suggested by Swamy and Thulasiraman.[3]

Similarity of graphs and enumeration of mutually dissimilar n-vertex labelled directed graphs

A graph $G(V, E)$, where V is the set of vertices and E is the set of edges, is a labelled graph if its vertices can be distinguished from one another by names such as $v_1, v_2, \ldots, v_n$. If G is labelled and each edge of G is assigned an orientation, then G is called an oriented or a directed labelled graph. G is said to be of order n if the number of vertices in G is equal to n. G is said to be of strength m if the number of edges in G is equal to m.

In G, (v_i, v_j) will denote the edge connecting the vertices v_i and v_j. Note that (v_i, v_j) and (v_j, v_i) refer to the same edge of G. It is assumed that there are no self-loops or parallel edgs in the graphs. If $A \subseteq V$, $B \subseteq V$, and $A \cap B = \phi$, the empty set, then

$$(A, B) = \{(v_i, v_j) \mid v_i \in A,\ v_j \in B,\ i \neq j\}.$$

For any set $A \subseteq V$, $\bar{A}$ will denote its complement in V. The set of edges $(A, \bar{A})$ is called a cut of the graph. For any set A, $|A|$ will denote its cardinality.

Consider an n^{th} order symmetric sign pattern $Y^{(s)}$ of strength m. $Y^{(s)}$ corresponds to a labelled directed graph $G(V, E)$ defined as follows:

$$|V| = n \text{ and } |E| = m.$$

Vertex v_i, $1 \leq i \leq n$ corresponds to row i of $Y^{(s)}$.

$(v_i, v_j) \in E$ if $y_{ij}^{(s)}$ is not equal to 0.

If $j > i$, then
(a) (v_i, v_j) is oriented from vertex v_i to vertex v_j, if $y_{ij}^{(s)} = +$.
(b) (v_i, v_j) is oriented from vertex v_j to vertex v_i, if $y_{ij}^{(s)} = -$.

$$Y^{(s)} = \begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

Figure 1a: *Sign pattern $Y^{(s)}$.*

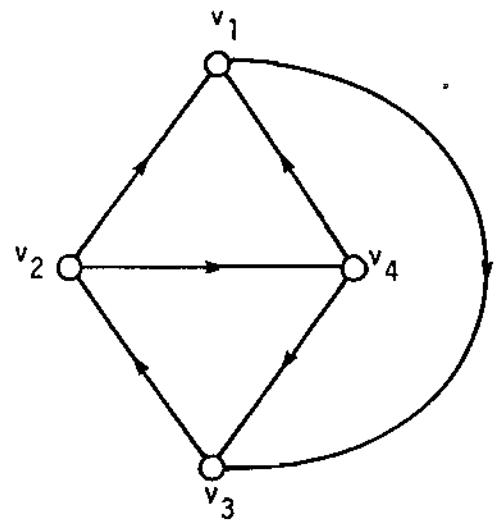

Figure 1b: *Graph G corresponding to $Y^{(s)}$.*

For example, the sign pattern $Y^{(s)}$ shown in Figure 1(a) corresponds to the labelled directed graph G shown in Figure 1(b). The graph $G^{(u)}$ obtained from a directed graph G after ignoring the orientations of the edges of G is called the undirected graph corresponding to G.

Consider now the set R of all n-vertex labelled connected directed graphs all of which have the same corresponding undirected graph $G^{(u)}$. Let S be a subset of the vertex set V of $G^{(u)}$. Then we define a function α_S: $R \rightarrow R$ as follows. If $G \in R$, then $\alpha_S(G) = G_S$, where G_S is the n-vertex labelled directed graph obtained from G after changing the orientations of all the edges incident on the vertices of the set S, one at a time. Note that $G_S \in R$.

For example, for the graph G shown in Figure 2(a), if $S = \{v_1, v_2, v_3\}$, then the graph G_S will be as shown in Figure 2(b). We note the following:

$$\alpha_S(G) = \alpha_{\bar{S}}(G)$$

$$\alpha_S(G_S) = G$$

and $\alpha_V(G) = \alpha_\phi(G) = G$

The composition $(\alpha_S \cdot \alpha_T)$ of two functions α_S and α_T is defined as:

$$(\alpha_S \cdot \alpha_T)G = \alpha_S(\alpha_T(G)) = \alpha_S(G_T).$$

Note that

$$\alpha_S \cdot \alpha_T(G) = \alpha_{S\theta T}(G) = \alpha_T\, \alpha_S(G)$$

where

$$S \theta T = S \cup T - (S \cap T).$$

If S and T are disjoint, then

$$\alpha_S \cdot \alpha_T(G) = \alpha_{S \cup T}(G).$$

Similarity of directed graphs

Two n-vertex labelled directed graphs $G^1(V, E)$ and $G^2(V, E)$ having the same corresponding undirected graph $G^{(u)}$ are similar if there exists a subset S of V such that $\alpha_S(G^1) = G^2$ i.e. $G^1_S = G^2$. Otherwise, G^1 and G^2 are dissimilar.

It can be seen that symmetric sign patterns are similar if the cor-

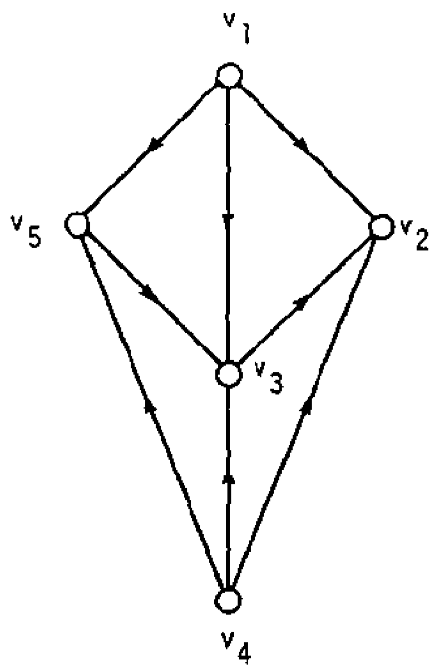

Figure 2a: *Graph G.*

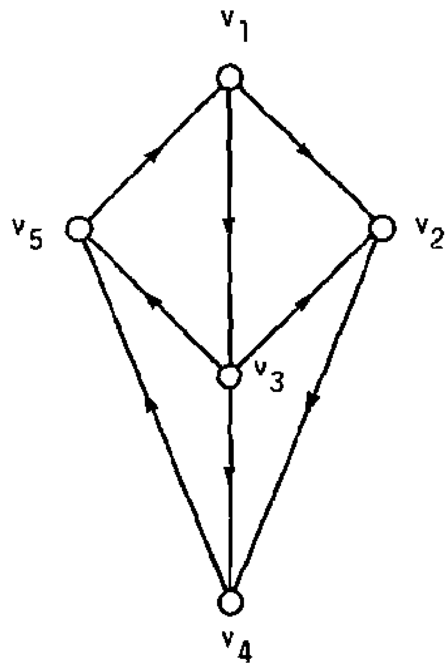

Figure 2b: *Graph G_s. $S = \{v_1, v_2, v_5\}$.*

responding directed graphs are similar. It can be shown that the similarity relation is an equivalence relation. We shall use the symbol '~' to denote this relation. Let $R = \{G^1, G^2, G^3, \ldots\}$. Let $R_1, R_2, \ldots, R_p$ be the p equivalence classes into which the set R is partitioned by the equivalence relation ~. then the fact $G^i \in R_k$ and $G^j \in R_k$ for some $i, j = 1, 2 \ldots, |R|$ implies that $G^i \sim G^j$. Also, if for $q \neq k$, $G^i \in R_q$ and $G^j \in R_k$, then G^i and G^j are dissimilar. Thus the maximum number of mutually dissimilar n-vertex labelled directed connected graphs in R is equal to the number of equivalence classes into which R is partitioned by the similarity relation. Note that

$$|R| = \sum_{i=1}^{p} |R_i|. \qquad (1)$$

If $G^{(u)}$ is of stength m, then $|R| = 2^m$. We now proceed to show that $|R_i| = 2^{n-1}, i = 1, 2, \ldots, p$.

Let, for any two labelled directed graphs $G^1(V, E) \in R$ and $G^2(V, E) \in R$, $E_d(G^1, G^2) \subseteq E$ *denote the set of edges which have* different orientations in G^1 and G^2. That is,

$$E_d(G^1, G^2) = \{(v_i, v_j) \in E \mid \text{orientation of } (v_i, v_j)$$

$$\text{in } G^1 \text{ is different from its orientation in } G^2\}.$$

Consider any two *similar graphs G* and G_s. The following can then be observed:

- If $v_i \in S(\bar{S})$ and $v_j \in S(\bar{S})$, then the orientation of edge (v_i, v_j) in G is the same as its orientation in G_s.
- If $v_i \in S$ and $v_j \in \bar{S}$, then the orientation of (v_i, v_j) in G is different from its orientation in G_s.

Thus the orientation of each edge $(v_i, v_j) \in (S, \bar{S})$ in G is different from its orientation in G_s. All other edges have the same orientation in both G nd G_s. Hence $E_d(G, G_s) = (S, \bar{S})$. Further, if for two graphs $G^1(V, E) \in R$ and $G^2(V, E) \in R$, $E_d(G^1, G^2) = (S, \bar{S})$ for some $S \subseteq V$, then $\alpha_s(G^1) = G^2$, and so, $G^1 \sim G^2$. Thus, we have the following theorems:

Theorem 1: Two labelled directed graphs $G^1(V, E) \in R$ and $G^2(V, E) \in R$ are similar iff for some $S \subseteq V$

$$E_d(G^1, G^2) = (S, \bar{S}).//$$

Theorem 2: Let $G(V, E)$ be a connectd graph. Let S and T be subsets of V such that neither S or $\bar{S}$ is equal to T. Then

$$(S, \bar{S}) \neq (T, \bar{T}).$$

Proof: Let

$$S = A \cup B$$

and $\quad T = A \cup C,$

where
$$A = S \cap T,$$
$$B = S - (S \cap T) = S \cap \bar{T}$$

and
$$C = T - (S \cap T) = T \cap \bar{S}$$

Let $\quad D = \bar{S} \cap \bar{T}.$

Note that A, B, C and D are disjoint subsets of V and their union is equal to V. Hence we have $\bar{S} = C \cup D$ and $\bar{T} = B \cup D$. Then

$$(S, \bar{S}) = (A \cup B, C \cup D)$$

$$= (A, C) \cup (A, D) \cup (B, C) \cup (B, D) \qquad (2)$$

and

$$(T, \bar{T}) = (A \cup C, B \cup D)$$

$$= (A, B) \cup (A, D) \cup (C, B) \cup (C, D). \qquad (3)$$

Contrary to the theorem, let

$$(S, \bar{S}) = (T, \bar{T}).$$

Then from Eqs. (2) and (3), we have

$$(A, C) = (B, D) = (A, B) = (C, D) = \phi,$$

the empty set. It therefore follows that

$$(A \cup D, C \cup B) = \phi.$$

Thus the vertices in $(A \cup D)$ are not connected to the vertices in $(C \cup B)$. Since $(A \cup D)$ is the complement of $(C \cup B)$, G is not connected. Since this contradicts the hypothesis, $(S, \bar{S}) \neq (T, \bar{T})$.//

It follows from Theorems 1 and 2 that every cut $(S, \bar{S})$ defines a unique graph G^j_S similar to G^j for any connected graph $G^j(V, E) \in R$. If $|V| = n$, then there are 2^{n-1} distinct cuts, if G^j is connected. Thus the graph G^j is similar to 2^{n-1} distinct graphs. Since all the graphs of the equivalence class R_i containing G^j are similar,

$$|R_i| = 2^{n-1};\ i = 1, 2, \ldots, p.$$

It then follows from Eq. (1) that

$$p = \frac{2^m}{2^{n-1}} = 2^{(m-n+1)}.$$

The following theorem is then derived.

Theorem 3: The maximum number of mutually dissimilar connected n-vertex labelled directed graphs having the same corresponding undirected graph $G^{(u)}$ of strength m is equal to $2^{(m-n+1)}$.//

Corollary 3.1: The maximum number of mutually dissimilar connected n-vertex labelled directed graphs having the same corresponding undirected graph $G^{(u)}$ of strength m is equal to 2^{m-n+c} where c is the number of connected components of $G^{(u)}$.// (Note that $m-n+c$ is cyclomatic number or nullity of the graph $G^{(u)}$).

We now proceed to develop an algorithm to generate a maximal set of mutually dissimilar lebelled directed graphs in R. Consider any two labelled directed graphs $G^1(V, E) \in R$ and $G^2(V, E) \in R$. Let $|V| = n$ and $|E| = m$. Let $T^{(u)} = (V, E_t)$ be any spanning tree of $G^{(u)}$, and

$$E_{t_d}(G^1, G^2) = E_d(G^1, G^2) \cap E_t.$$

Note that if $T^1(V, E_t)$ is the subgraph of $G^1(V, E)$ corresponding to the non-oriented tree $T^{(u)}$, and $T^2(V, E_t)$ is the subgraph of $G^2(V, E_t)$ corresponding to $T^{(u)}$, then $E_{t_d} = E_d(T^1, T^2)$.

Theorem 4: If $E_d(G^1, G^2) \neq \phi$ and $E_{t_d}(G^1, G^2) = \phi$ for some spanning tree $T^{(u)}$ of $G^{(u)}$, then G^1 and G^2 are dissimilar.

Proof: It is known that a cut of a graph contains at least one edge of every spanning tree.[3] Since $E_{t_d}(G^1, G^2) = \phi$, it follows that no edge of $T^{(u)}$ is in $E_d(G^1, G^2)$. Hence, $E_d(G^1, G^2)$ is not a cut of $G^{(u)}$. Therefore, from Theorem 1, G^1 and G^2 are dissimilar.//

Consider again the set $R = \{G^1, G^2, \ldots\}$. Let the edges of $G^{(u)}$ be denoted as $e_1, e_2, \ldots, e_m$. Let for each k, $e_k = (v_a, v_b)$, $b > a$. We define an m-digit binary sequence b^j for each element G^j of R as

$$b^j = [b^j_1\ b^j_2 \ldots b^j_m],$$

where

$$b^j_k = \begin{cases} 1, \text{ if the orientation of } e_k \text{ in the graph } G^j \text{ is from } v_a \text{ to } v_b. \\ 0, \text{ if the orientation is from } v_b \text{ to } v_a. \end{cases} \qquad (4)$$

Note that each graph G^j is uniquely represented by b^j.

Let $E_t = \{e_{k_1}, e_{k_2}, \ldots, e_{k_{n-1}}\}$ be a set of $(n-1)$ edges of some spanning tree $T^{(u)}$ of $G^{(u)}$. Construct a set of $2^{(m-n+1)}$ distinct binary sequences $b^1, b^2, \ldots, b^p$ where $p = 2^{(m-n+1)}$ such that

$$b^i_{k_r} = b^j_{k_r}, \text{ for all } i, j = 1, 2, \ldots, p;\ i \neq j;$$

$$r = 1, 2, \ldots, n-1.$$

That is,

$$E_{t_d}(G^i, G^j) = \phi \text{ for all } i, j = 1, 2, \ldots, p,\ i \neq j.$$

Hence by Theorem 4 these p graphs represent a maximal set of mutually dissimilar n-vertex labelled directed graphs in R. The following algorithm summarizes this procedure.

Algorithm 1:

- Name the edges of $G^{(u)}$ as $e_1, e_2, \ldots, e_m$.
- Choose any spanning tree $T^{(u)}$ of $G^{(u)}$ such that

$$E_t = \{e_{k_1}, e_{k_2}, \ldots, e_{k_{n-1}}\}$$

- Generate $p = 2^{(m-n+1)}$ distinct binary sequences $b^1, b^2, \ldots, b^p$ such that $b^i_{k_r} = b^j_{k_r}$ for all $i, j = 1, 2, \ldots, p$, $i \neq j$;

$$r = 1, 2, \ldots, n-1.$$

- Generate the p labelled directed graphs represented by these p sequences. These graphs $G^1, G^2, \ldots, G^p$ represent a maximum set of mutually dissimilar n-vertex labelled directed connected graphs having the same corresponding undirected graph $G^{(u)}$ of strength m.

Consider the undirected graph $G^{(u)}$ shown in Figure 3. Choose the spanning tree consisting of the edges e_1, e_4, e_6 and e_7. Here $m = 7$ and $n = 5$.
Therefore $p = 2^3 = 8$. The binary sequences $b^1, b^2, \ldots, b^8$ are as follows:

	e_1	e_2	e_3	e_4	e_5	e_6	e_7
$b^1 =$	1	0	0	0	0	1	1
$b^2 =$	1	0	0	0	1	1	1
$b^3 =$	1	0	1	0	0	1	1
$b^4 =$	1	0	1	0	1	1	1
$b^5 =$	1	1	0	0	0	1	1
$b^6 =$	1	1	0	0	1	1	1
$b^7 =$	1	1	1	0	0	1	1
$b^8 =$	1	1	1	0	1	1	1

The graphs $G^1, G^2, \ldots, G^8$ are then obtained using Eq. (4). For example, the graphs G^3 and G^7 are as shown in Figure 4(a) and Figure 4(b) respectively.

An algorithm for testing similarity of graphs

In this section an algorithm to test similarity of two n-vertex labelled directed graphs is developed.

Theorem 5: If $T^1(V, E_t)$ and $T^2(V\ E_t)$ are two oriented labelled spanning trees having the same corresponding undirected graph $T^{(u)}$, then $T^1 \sim T^2$.

Proof: Consider the set R_t of all n-vertex oriented labelled spanning

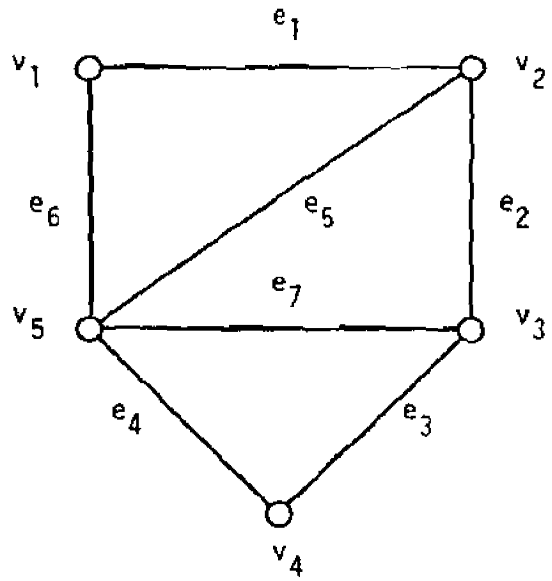

Figure 3: *Undirected graph $G^{(u)}$.*

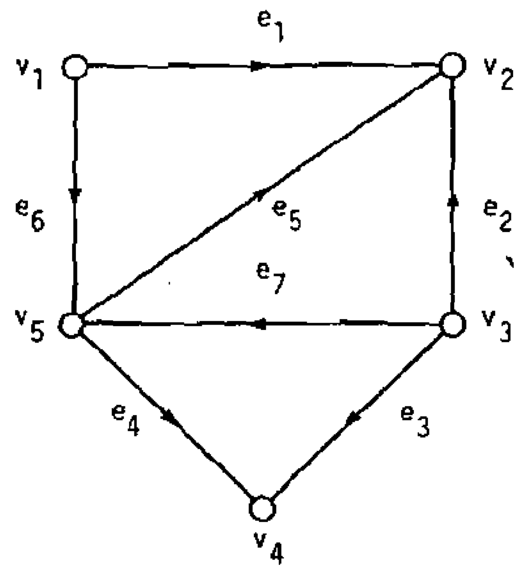

Figure 4a: *Graph G^1. b^1 = 1010011.*

trees having the same corresponding non-oriented tree $T^{(u)}$. If $|V| = n$, then $|E_t| = n - 1$. So, by Theorem 3, the number of equivalence classes into which R_t is partitioned by the similarity relation is equal to 1. Thus all the trees in R_t are similar.//

Let $T(V, E_t)$ be a spanning tree of a graph $G(V, E)$. If $S \subseteq V$, then let $(S, \bar{S})_t = (S, \bar{S}) \cap E_t$. That is, $(S, \bar{S})_t$ consists of those edges of $(S, \bar{S})$ that belong to E_t.

Consider now two labelled directed connected graphs $G^1(V, E)$ and $G^2(V, E)$ in R. Let $T^{(u)}$ be a spanning tree of $G^{(u)}$. Let $T^1(V, E_t)$ be the oriented labelled spanning tree of G^1 corresponding to $T^{(u)}$. T^2 is similarily defined. Then $E_d(T^1, T^2) = E_{d_t}(G^1, G^2)$. In view of Theorem 5, a subset $S \subseteq V$ exists such that $E_{d_t}(G^1, G^2) = (S, \bar{S})_t$. In addition, because of Theorem 2, S is unique. Note that if $E_{d_t}(G^1, G^2) = (S, \bar{S})_t$ then $\alpha_s(T^1) = T^2$.

Theorem 6: Consider two labelled connected directed graphs $G^1(V, E)$ and $G^2(V, E)$ having the same corresponding undirected graph $G^{(u)}$. Let $T^{(u)}$ (V, E_t) be a spanning tree of $G^{(u)}$. Let T^1 and T^2 be the oriented labelled spanning trees of G^1 and G^2 respectively, corresponding to $T^{(u)}$ and $E_{d_t}(G^1, G^2) = (S, \bar{S})_t$. Then $G^1 \sim G^2$ if and only if $E_d(G^1, G^2) = (S, \bar{S})$.
Proof: Sufficiency is obvious.
Necessity: Consider the graphs G_s^1 and G^2. Then

$$\alpha_s(G^1) = G_s^1.$$

and

$$\alpha_s(T^1) = T_s^1.$$

Since

$$E_{d_t}(G^1, G^2) = (S, \bar{S})_t,$$

$$\alpha_s(T^1) = T_s^1 = T^2.$$

Hence in the graphs G_s^1 and G^2 the oriented labelled spanning trees corresponding to $T^{(u)}$ are identical. That is, $E_{d_t}(G_s^1, G^2) = \phi$. If $G_s^1 \neq G^2$, i.e., $E_d(G_s^1, G^2) \neq \phi$, then by Theorem 4, G_s^1 is not similar to G^2 and so G^1 and G^2 are not similar. However, this contradicts the hypothesis, therefore $G_s^1 = G^2$, i.e., $\alpha_s(G^1) = G^2$. Therefore $E_d(G^1, G^2) = (S, \bar{S})$.//

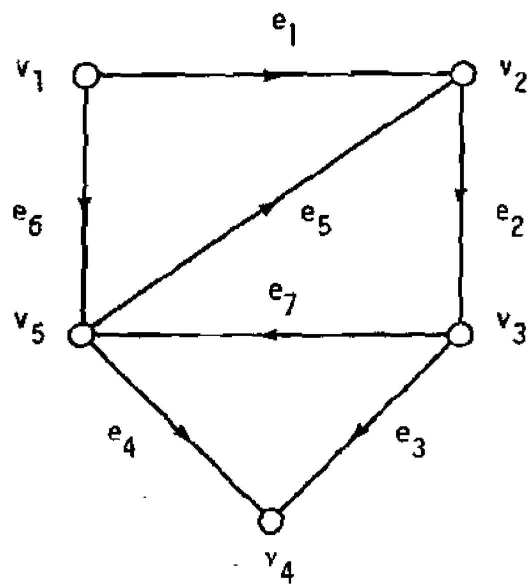

Figure 4b: *Graph G^2. b^2 = 1110011.*

The following observations are now made. By Theorem 1 the problem of testing similarity of two graphs G^1 and G^2 having the same corresponding undirected graph $G^{(u)}$ is equivalent to the problem of testing whether the set $E_d(G^1, G^2)$ is a cut of $G^{(u)}$. If $T^{(u)}$ is a spanning tree of $G^{(u)}$ and $E_{d_t}(G^1, G^2) = (S, \bar{S})_t$, then by Theorem 6 the problem of testing similarity of G^1 and G^2 reduces to that of checking whether $E_d(G^1, G^2) = (S, \bar{S})$.

The following are the main steps in testing similarity of G^1 and G^2:

- Obtain $E_d(G^1, G^2)$.
- Choose any spanning tree $T^{(u)}$ of $G^{(u)}$. Obtain $E_{d_t}(G^1, G^2) = E_t \cap E_d(G^1, G^2)$ where E_t is the set of edges of $T^{(u)}$.
- Obtain the unique $S \subseteq V$ where V is the set of vertices of $G^{(u)}$ such that $E_{d_t}(G^1, G^2) = (S, \bar{S})_t$.
- If $(S, \bar{S}) = E_d(G^1, G^2)$, then $G^1 \sim G^2$; otherwise they are dissimilar.

It may be seen from the above that the third step is crucial, i.e., to determine $S \subseteq V$ so that $E_{d_t} = (S, \bar{S})_t$.

The following two rules can be used to determine S:

- If $(v_i, v_j) \in E_d$, then either $v_i \in S$ and $v_j \in \bar{S}$ or $v_i \in \bar{S}$ and $v_j \in S$.
- If $(v_i, v_j) \notin E_d$, then either $v_i, v_j \in S$ or $v_i, v_j \in \bar{S}$.

hese rules are incorporated in Algorithm 2 to determine whether the set $E_d(G^1, G^2)$ is a cut of the graph $G^{(u)}$. The main feature of the algorithm is that it generates a spanning tree $T^{(u)}$ of $G^{(u)}$, and as the tree is generated the vertex sets S and $\bar{S}$ are also formed. At the end of the process of generation of $T^{(u)}$ the vertex set S will be known. The algorithm assigns a common label 1 to the vertices of the set S and a common label 2 to those of $\bar{S}$.

Algorithm 2:

- Assign label 1 to vertex v_1.
- Scan vertex v_1 and assign labels to all vertices adjacent to vertex v_1 according to the two rules above. At the end of this step vertex v_1 is said to be labelled and scanned.
- Scan the most recently labelled but unscanned vertex, e.g. v_j, and assign labels to all the unlabelled vertices adjacent to v_j according to the above rules. Now v_j is said to be labelled and scanned.
- Repeat the third step until all the vertices are labelled and scanned.
- The set S then consists of all the vertices which are assigned the label 1.
- If $E_d = (S, \bar{S})$, then E_d is a cut of graph $G^{(u)}$. Otherwise it is not a cut.

Conclusions

In this paper the concepts of similarity of labelled directed graphs and similarity of n^{th} order symmetric sign patterns are introduced. We have shown that each n^{th} order symmetric sign pattern corresponds to an n-vertex labelled directed graph. We have also shown that two sign patterns are similar iff their corresponding directed graphs are similar. The following results have been established:

- Necessary and sufficient conditions for two graphs to be similar (Theorem 1);
- The maximum number of mutually dissimilar n-vertex labelled directed graphs (Theorem 3);
- An algorithm for generating a set of mutually dissimilar n-vertex labelled directed graphs (Algorithm 1); and
- An algorithm for testing the similarity of two graphs (Algorithm 2).

These results provide answers to the corresponding questions concerning sign patterns.

References

1. Biorci, G., "Sign matrices and the realizability of conductance matrices," IEE (London), Mono. No. 424 E, December 1960.
2. Naidu, M.G.G., Reddy, P.S. and Thulasiraman, K., "Continuously equivalent realizations of third-order paramount matrices," *Int'l J. of Circuit Theory and Applications*, Vol. 5, 1977, pp. 403–408.
3. Swamy, M.N.S. and Thulasiraman, K., *Graphs, Networks and Algorithms*, Wiley-Interscience, 1981.