On a Theorem in Graph Theory

An error in the proof of a theorem given in Ref. 1 is pointed out and a correct proof is provided.

Seshu and Reed[1] have proved the following theorem in their book[1]:

If G is a connected graph of v vertices and G_s is a subgraph of G with $v-1$ elements and containing no circuits, then G_s is a tree of G.

In this correspondence, we first point out an error in the proof of the above theorem and then provide a correct proof.

This theorem will be true if it can be shown that:

(1) G_s contains v vertices, and

(2) G_s is connected.

In their proof, Seshu and Reed first state that G_s contains all the v vertices of G and then establish (2). They then argue that since G_s is connected, contains $v-1$ elements and no circuits, it is its own tree and hence contains v vertices. Thus they establish (2) assuming (1) to be true and then establish (1) assuming (2) to be true. Since (1) and (2) should be proved independently, the proof as given in Ref. 1 is not valid. The following comments are also in order:

(i) by definition [def. 1–4 in Ref. 1], a subgraph G_s of a connected graph G consists of a subset of edges (and hence also the vertices at which these edges are incident) of G. Thus according to this definition G_s need not, in general, contain all the vertices of G. Hence equation (2-5) used in Ref. 1 is not correct.

(ii) Suppose we define that a subgraph G_s of a graph G consists of all the vertices of G and a subset of edges of G. Then equation (2-5) is correct. However, in such a case, the statements 'Now G_s is its own tree and contains $v-1$ elements. Hence G_s contains v vertices . . . are not necessary. This latter definition of a subgraph is not the usual one. Such a definition, though convenient for the proof of this theorem, will lead to difficulties in the development of other results of graph theory.

Thus a correct proof of the theorem is called for. This is detailed as follows: Let G_s consist of p maximal connected subgraphs. Let $s_1, s_2, s_3, \ldots, s_p$ be the number of vertices in s_i. Since s_i is connected and contains no circuits s_i is its own tree. Hence s_i contains $v_i - 1$ elements. Thus G_s contains $v^* - p$ elements where

$$v^* = \sum_{i=1}^{p} v_i \quad .$$

Since, by hypothesis, G_s contains $v-1$ elements, we get

$$v^* - p = v - 1 \qquad (1)$$

Hence

$$v^* - v = p - 1 \qquad (2)$$

However, $v^* \leqslant v$ and hence we get from (2)

$$p^* \leqslant 1 \tag{3}$$

But

$$p \geqslant 1 \tag{4}$$

Hence from (3) and (4) it follows that $p = 1$. Thus G_s is connected.

Since $p = 1$ we get from (1)

$$v^* = v$$

Hence G_s contains all the v vertices. Thus follows the proof of the theorem.

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Reference

1. *S. Seshu and M. B. Reed* Linear Graphs and Electrical Networks, p. 26, Addison Wesley Publishing Company, Reading, Massachusetts, U.S.A.

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