K-SETS OF A GRAPH AND VULNERABILITY OF COMMUNICATION NETS

by

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The concept of the k-vulnerability of communication nets is introduced and a procedure to design k-invulnerable communication nets is given. In the course of this study several important properties of the k-sets of a graph are discussed. Also the problem of generating the k-sets is elucidated. Illustrative examples are worked out.

INTRODUCTION

In the recent past several results relating to the vulnerability of communication nets have appeared in the literature. Graph theory has played an important role in establishing these results. Vulnerability studies have been based on suitable vulnerability criteria, which in turn are related to certain relevant concepts in graph theory. A detailed exposition of these topics is available in Ref. 1. In the present paper we study the vulnerability of communication nets based on the properties of what will be called the K-sets of a graph.

We first introduce the notation to be followed. $G = (V, E)$ will represent a graph with v-vertices and e-edges. V and E will denote the set of vertices and the set of edges respectively. $|X|$ will denote the number of elements in the set X. For any two vertices v_i and v_j, we define

$$< v_i, v_j > = 1, \text{ if } v_i \text{ and } v_j \text{ are adjacent}$$

$$= 0, \text{ otherwise.}$$

$X \subseteq V$ is said to be independent, if

$$< x_i, x_j > = 0, \ \forall \ x_i, x_j \in X$$

The complement in V of a subset X of V will be denoted by $\overline{X}$.

$[x]$ will denote the largest integer less than or equal to x.

$[x]^*$ will denote the smallest integer greater than or equal to x.

(v_i, v_j) will denote the edge connecting vertices v_i and v_j.

(V_1, V_2) will denote all the edges (v_i, v_j) where $v_i \in V_1$ and $v_j \in V_2$.

A graph is said to be p-regular if each vertex in G is of degree p.

A vertex of v_x of G is said to be *stripped* if all the edges incident on v_x is removed from G.

A graph which is in the form of a tree will be denoted by G_T.

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DEFINITION AND PROPERTIES OF K-SETS OF A GRAPH

Definition 1: A *K-set* of a graph G is a *minimal* set of vertices of G such that the rank of the graph which results after stripping all the vertices in this set is equal to zero.

We note that the graph resulting after all the vertices in a K-set are stripped will contain only isolated vertices.

Example 1: For the graph shown in Fig. 1, it may be verified that v_a, v_b, v_d, v_e and v_a, v_c, v_e are K-sets. It may be noted that there may be some more K-sets for G.

Denoting by $S_k = \{K_i\}$ the collection of all K-sets of G, let $k_{\min} = \underset{K_i \in S_k}{\text{Min}} \{ |K_i| \}$.

A K-set containing $k_{\min}$ vertices will be called a $K_{\min}$-*set.*

It may be observed that a $K_{\min}$ set is not unique.

For graph G shown in Fig. 1. $k_{\min} = 3$ and the set $\{v_a, v_c, v_e\}$ is a $K_{\min}$-set.

It may be noted that a $K_{\min}$-set of a graph G is also referred to as a point cover [1].

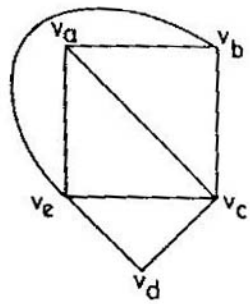

Fig. 1

Definition 2: An *L-set* of a graph G is a *maximal* set of independent vertices of G.

For G shown in Fig. 1, it may be verified that $\{v_b, v_d\}$ and $\{v_a, v_d\}$ are L-sets. It should be noted that the vertex v_c is by itself an L-set. Denoting by $S_L = \{L_i\}$, the collection of all L-sets of G, let

$$l_{\max} = \underset{L_i \in S_L}{\text{Max}} \{ |L_i| \} .$$

An L-set containing $l_{\max}$ vertices will be called an $L_{\max}$-*set.*

Theorem 1: A subset of V is a K-set if its complement is an L-set.

Proof: *Necessity*

Let a set $X \subset V$ be a K-set. It easily follows from the definition of a K-set that $\overline{X}$ is a set of independent vertices. We next prove, by contradiction, that $\overline{X}$ is a maximal set of independent vertices.

If $\overline{X}$ is not a maximal set then X_1 denotes a maximal subset of X such that

$$< x_i, x_j > = 0 \quad x_i, x_j \in X_1$$

and

$$< x_i, x_j > = 0 \quad x_i \in X_1 \ , \ \ x_j \in \overline{X}$$

It then follows that $(X - X_1)$ is a K-set. This is however a contradiction, since no proper subset of a K-set is a K-set. Hence the necessity theorem.

Sufficiency

Let a set $X \subset V$ be an L-set. Since every edge of G is incident on at least one vertex in $\overline{X}$, it follows that $\overline{X}$ must contain a K-set. We next show by contradiction that no proper subset of $\overline{X}$ is a K-set, i.e., $\overline{X}$ is itself a K-set.

Let $(\overline{X})_1 \subset \overline{X}$ be a K-set. It then follows from the necessity part of the theorem that $X \cup \overline{X} - (\overline{X})_1$ is an L-set. This implies that X is not a maximal set of independent vertices. This again contradicts the hypothesis. Hence the sufficiency.

We next discuss some properties of K-sets of a graph.

Property 1: If a graph G is a Lagrangean tree then $k_{\min} = 1$.

The above property can easily be verified to be true.

Property 2: If a graph G is a linear tree then $k_{\min} = \left[\frac{v}{2}\right]$.

Proof: Let the vertices of G be numbered consecutively starting from a top vertex. Then all the odd numbered vertices will constitute an $L_{\max}$-set. The total number of odd numbered vertices is equal to $\left[\frac{v}{2}\right]^*$. Hence

$$k_{\min} = v - \left[\frac{v}{2}\right]^*$$

$$= \left[\frac{v}{2}\right]$$

Properties 3, 4, 5 follow easily from property 2.

Property 3: If a graph G contains a path with p-vertices, then $k_{\min} \geqslant \left[\frac{p}{2}\right]$.

Property 4: If a graph G contains r vertex disjoint paths, then

$$k_{\min} \geqslant \sum_{i=1}^{n} \left[\frac{p_i}{2}\right]$$

where p_i is the number of vertices in the ith path.

Property 5: If a graph contains a linear tree, then $k_{\min} > \left[\frac{v}{2}\right]$.

Property 6: For a complete bi-partite graph G for which $E = (V_1, V_2)$,

$$k_{\min} = \min\{|V_1|, |V_2|\} .$$

The above property can easily be verified to be true.

Property 7: For a v-vertex p-regular graph $k_{\min} \geqslant \text{Max}\left\{p, \left[\frac{v}{2}\right]^*\right\}$.

Proof: Consider any K-set of a v-vertex, p-regular graph G. Let v_i be a vertex not present in the K-set. To isolate the vertex v_i all the edges incident on v_i should be removed. This is possible only if all the vertices adjacent to v_i are present in the K-set. Since each vertex is of degree p, $|K| > p$. Hence

$$k_{\min} \geqslant p .$$

Next we show that $k_{\min} > \left[\frac{v}{2}\right]^*$. The total number of edges in G is equal to $p\frac{v}{2}$. Since the maximum number of edges that can be removed by stripping any vertex in a K-set is p, it follows that a K-set must contain at least

$$\left[p\frac{v}{2}\cdot\frac{1}{p}\right]^{*} = \left[\frac{v}{2}\right]^{*} \text{ vertices}$$

This proves the property.

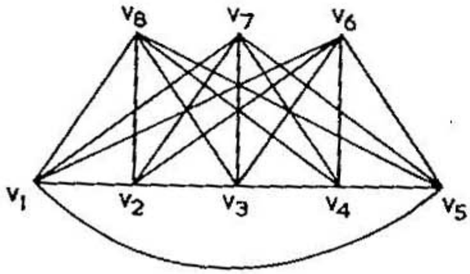

Fig. 2(a)

$$p = 5 > \left[\frac{v}{2}\right]^{*} \quad k_{\min} = 5$$

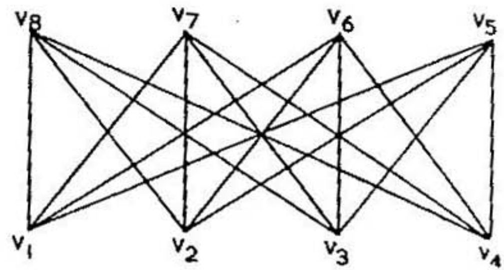

Fig. 2(b)

$$p = 4 = \left[\frac{v}{2}\right]^{*} = k_{\min}$$

Three graphs are shown in Fig. 2 to demonstrate the existence of regular graphs whose $k_{\min}$ coincides with the lower bound given in property 7. Hence this lower bound cannot be improved upon.

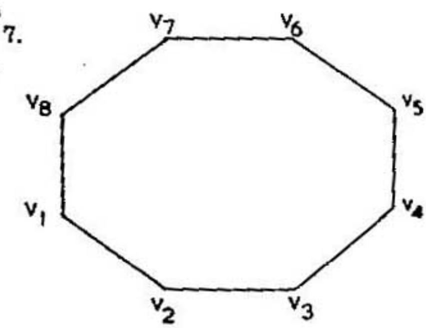

Fig. 2(c)

$$p = 2 < \left[\frac{v}{2}\right]^{*} = 4 = k_{\min}$$

Theorem 2: For a v-vertex graph G_T,

$$k_{\min} \leqslant \left[\frac{v}{2}\right].$$

Proof: Let v_{1a} be a tip vertex of G_T and v_{1b} adjacent to v_{1a}. The graph G_1 resulting after stripping v_{1b} will contain at least two isolated vertices. Choose then a tip vertex v_{2a} of G_1. The graph G_2, resulting after stripping the vertex v_{2b} adjacent to v_{2a}, will contain at least four isolated vertices.

Let G_i denote the graph which results after the ith operation. It may be seen that G_i will contain at least $2i$ isolated vertices.

Let G_n contain only isolated vertices. Then $\{v_{1b}, v_{2b}, \ldots, v_{nb}\}$ will be a K-set of G_T. Further $2n \leqslant v$ or $n \leqslant \left[\frac{v}{2}\right]$. Hence

$$k_{\min} \leqslant \left[\frac{v}{2}\right].$$

(To be continued)