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ON AN EXTREMAL PROBLEM IN GRAPH THEORY AND ITS APPLICATION

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I. INTRODUCTION

The starting point for extremal problems in graph theory was the work of Turan [1,2]. Some of the results available in this area of graph theory can be found in [3] and [4]. Solution of extremal problems finds application in the design of optimally invulnerable communication nets. An approach usually followed in the vulnerability studies of communication nets is to define a meaningful vulnerability criterion and then relate the design of optimally invulnerable (with respect to the chosen criterion) communication nets, to the design of v-vertex e-edge graphs having a specified property. This approach has been followed in many of the results given in [5]. The results reported in [6] and in the present paper have been motivated by this consideration.

In [6], the concept of k-vulnerability of communication nets was introduced and the design of optimally k-invulnerable communication nets related to the design of v-vertex e-edge graphs having the largest possible point covering number.

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A new proof of Turan's result [7] on the minimum number of edges required to realise a v-vertex graph having a specified point covering number was given. An upper bound on the point covering number for a graph having v-vertices and e-edges was also established. In this paper, we consider the following dual problems:

1. Identification of maximum-edge v-vertex graphs having a specified edge independence number and having specified incidence relationship between matched and unmatched vertices.
2. Design of v-vertex e-edge graphs having the smallest edge independence number.

As we discuss the above problems we also establish a theorem due to Erdos and Gallai [8].

We now introduce the notation that will be followed in the paper.

G(V,E) will denote an undirected graph without parallel edges and self loops, where V is the set of vertices and E is the set of edges of the graph. (v_i, v_j) will denote the edge connecting vertices v_i and v_j. Thus $E \subseteq V \times V$.

The function f_g: $V X V \rightarrow \{0,1\}$ will be defined as follows:

$$f_g(v_i, v_j) = 1, \text{ if } (v_i, v_j) \in E \text{ and } v_i \neq v_j$$
$$= 0, \text{ otherwise.}$$

If S and T are mutually disjoint subsets of V, then

$$(S,T) = \{(v_i , v_j) \mid v_i \in S,\ v_j \in T,\ v_i \neq v_j\} .$$

Then we define

$$f_g(S,T) = 1, \quad \text{if} \quad (S,T) \subseteq E$$

$$= 0, \quad \text{if} \quad (S,T) \cap E = \emptyset \ \text{the null set.}$$

For any set X, $|X|$ will denote the cardinality of X.

A set of edges in a graph is independent, if no two of them are adjacent.

The edge independence number p_{max} of a graph is the largest number of edges in any independent set of the graph.

An independent set of p_{max} edges of a graph is called a *maximum matching* of the graph.

The above concepts are discussed in [4], [7].

II. MAXIMUM-EDGE v-VERTEX GRAPHS HAVING A SPECIFIED p_{max}

Let $G(V,E)$ be a v-vertex graph with $p_{max} = p$. Let $\{e'_1, e'_2, \ldots, e'_p\}$ be a set of p independent edges in $G(V,E)$, where

$$e'_i = (a_i , b_i), \ i = 1, 2, \ldots, p.$$

Let

$$A = \{a_1, a_2 , \ldots, a_p\}$$

and

$$B = \{b_1, b_2, \ldots, b_p\}$$

then the set $(A \cup B)$ will represent the vertex set of the edges $e'_1, e'_2, \ldots, e'_p$. Let $V_b = V - (A \cup B)$. We now define a partition $\pi = \{V_0, V_1, V_2, V_0^*, V_1^*, V_2^*\}$ of the set $(A \cup B)$ according to the following rules:

i) Let a_i be not adjacent to any vertex in V_b. Then

   a) $a_i \in V_o$ and $b_i \in V_o^*$ if b_i is not adjacent to any vertex in V_b.

   b) $a_i \in V_1^*$ and $b_i \in V_1$ if b_i is adjacent to exactly one vertex in V_b.

   c) $a_i \in V_2^*$ and $b_i \in V_2$ if b_i is adjacent to two or more vertices in V_b.

ii) $a_i \in V_1$, and $b_i \in V_1^*$ if a_i is adjacent to exactly one vertex in V_b.

iii) $a_i \in V_2$, $b_i \in V_2^*$ if a_i is adjacent to two or more vertæces in V_b.

Let, without loss of generality,

$$V_2 = \{v_1, v_2, \ldots, v_x\}, \qquad |V_2| = x$$

$$V_1 = \{v_{x+1}, v_{x+2}, \ldots, v_{x+y}\}, |V_1| = y$$

$$V_0 = \{v_{x+y+1}, v_{x+y+2}, \ldots, v_p\}, |V_0| = p - (x+y)$$

$$V_2^* = \{v_1^*, v_2^*, \ldots, v_x^*\}$$

$$V_1^* = \{v_{x+1}^*, v_{x+2}^*, \ldots, v_{x+y}^*\}$$

and

$$V_0^* = \{v_{x+y+1}^*, v_{x+y+2}^*, \ldots, v_p^*\}$$

Let $e_i = (v_i, v_i^*)$, $i = 1,2, \ldots, p$. It may be seen that the set of p edges $e_1, e_2, \ldots, e_p$ is only a permutation of the set $\{e'_1, e'_2, \ldots, e'_p\}$. Hence $e_1, e_2, \ldots, e_p$ are also independent.

It may be observed that an I-pair with respect to v_i and v_j along with the set of (p-2) independent edges $\{e_k \mid k = 1,2,\dots, p, \ k \neq i, \ k \neq j\}$ forms a set of p independent edges. This observation plays a significant role in the proofs of many of the lemmas that follow.

Through a series of lemmas, we now investigate the nature of the function f_g. That is, we investigate whether a subset of (VXV) is also a subset of E in G(V,E). Unless stated otherwise, all the discussions that follow are with respect to the graph G(V,E) defined at the beginning of this section. This graph will be referred to simply as G.

<u>Lemma 1</u>:

$$f_g(V_b, V_0^*) = f_g(V_b, V_0) = 0.$$

<u>Proof</u>: This follows from the definition of V_0 and V_0^*. //

<u>Lemma 2</u>:

i) $f_g(V_b, V_b) = 0$

ii) $f_g(V_2^*, V_2^*) = 0$

iii) $f_g(V_2^*, V_1^*) = 1$

iv) $f_g(V_b, V_2^*) = 0$

<u>Proof</u>: We prove the lemma by contradiction.

i) Let, for some $v_i, v_j \in V_b$
$f_g(v_i, v_j) = 1$.
Then it may be seen that the (p+1) edges
$(v_i, v_j), e_1, e_2, \dots, e_p$ will form an independent set.

is contradicts the assumption: that for G, $p_{max} = p$. (3)

t $f_g(v_i^*, v_j^*) = 1$, for some $v_i^*, v_j^* \in V_2^*$. Then

e (p-1) edges (v_i^*, v_j^*), $e_1, e_2, \ldots, e_{i-1}, e_{i+1}, e_{i+2}$, (4)

.., $e_{j-1}, e_{j+1}, e_{j+2}, \ldots, e_p$ along with an I-pair of

and v_j will form an independent set of (p+1) edges, (5)

ntradicting the assumption that $p_{max} = p$ for G. (6)

t for some $v_i^* \in V_2^*$ and $v_j^* \in V_1^*$, $f_g(v_i^*, v_j^*) = 1$.

e proof then proceeds in exactly the same way as in (7)

i) above.

for some $v_i \in V_b$ and $v_j^* \in V_2^*$, $f_g(v_i, v_j^*) = 1$, (8)

en the (p+1) edges (v_i, v_j^*), $e_1, e_2, \ldots, e_{j-1}, e_{j+1}$,

.., e_p and (v_j, v_1) where $v_1 \in V_b (v_1 \neq v_i)$, will form

independent set leading to a contradiction.//

]:

y(v-2p-1) of the edges in the set (V_b^*, V_1^*) will not that

ent in G.

By definition, each vertex in V_1 is connected to which

only one vertex in V_b. Thus only y of the y(v-2p)

n the set (V_b, V_1) will be present in G.//

et n_c denote the number of edges in a v-vertex

e graph, i.e.,

$$n_c = \frac{v(v-1)}{2} \quad (1)$$

that

$$|(V_b, V_b)| = \frac{(v-2p)(v-2p-1)}{2} = n_1 \quad (2)$$

Proof:

i) Let contrary to the lemma $f_g(v_i\ ,\ v_k^*) = 1$, for some $v_k^* \in (V_2^* \cup V_1^*)$, $v_k^* \neq v_j^*$. Then it may be seen that the (p-1) edges $(v_i^*\ ,\ v_j^*)$, $(v_i\ ,\ v_k^*)$, $e_1, e_2, \ldots, e_{j-1}, e_{j+1}, \ldots, e_{k-1}, e_{k+1} \ldots e_{i-1}, e_{i+1}, \ldots, e_p$ along with an I-pair of v_j and v_k will form a set of (p+1) independent edges. Hence a contradiction.

ii) From (i) above,

$$f_g(v_i\ ,\ v_k^*) = 0, \forall v_k^* \in V_2^* \cup V_1^*, \quad v_k^* \neq v_j^* \quad \text{and}$$

$$f_g(v_i\ ,\ v_k^*) = 0, \forall v_k^* \in V_2^* \cup V_1^*, \quad v_k^* \neq v_l^*$$

Hence

$$f_g(v_i\ ,\ v_k^*) = 0, \forall v_k^* \in V_2^* \cup V_1^*$$

i.e.,

$$f_g(v_i, V_2^* \cup V_1^*) = 0$$

Hence the proof. //

Threeom 2:

$$n_{x,y} \leqslant n_c - n_a - xz, \quad \text{if} \quad x + y \geqslant 2, \quad x \neq 0.$$

Proof:

Let z_i, $i = 1,2,\ldots, l$ be the number of vertices in V_0^* which are adjacent to i vertices in V_2^*. Then it follows from theorem (1) and lemma (4) that

$$n_{x,y} \leqslant n_c - n_a - \sum_{i=1}^{l} z_i(x-i) - z_1(x+y-1) - \sum_{i=2}^{l} z_i(x+y)$$

$$- (z - z_1 - z_2, \ldots, z_l)x$$

$$= n_c - n_a - zx - z_1(x+y-2) - \sum_{i=2}^{l} z_i(x+y-i)$$

$$\leqslant n_c - n_a - zx \quad //$$

Lemma 5:

$$f_g(v_i, v_j) = 0, \quad v_i \in V_b, \quad v_j \in V_1$$

$$\Longrightarrow \quad f_g(v_i , v_j^*) = 0.$$

Proof:

If contrary to the lemma $f_g(v_i , v_j^*) = 1$, then it may be seen that the (p+1) edges e_1, e_2, ..., e_{j-1}, e_{j+1}, ..., e_p, (v_i, v_j^*), (v_j, v_l) where $v_l \in V_b$, will form an independent set contradicting the assumption that $p_{max} = p$ for G. //

Lemma 6:

$$f_g(v_i, v_j^*) = 1, \quad v_i \in V_b, \quad v_j^* \in V_1^*$$

$$\Longrightarrow \quad f_g(v_j , v_i) = 1.$$

Proof:

If $f_g(v_j, v_i) = 0$, then there exists a $v_k \in V_b$ such that $f_g(v_j, v_k) = 1$. Hence the (p+1) edges (v_i, v_j^*), (v_j, v_k), e_1, e_2, ..., e_{j-1}, e_{j+1}, ..., e_p will form an independent set contradicting the fact that $p_{max} = p$ for G.//

Lemma 7:

$$f_g(v_i , v_j^*) = 1, \quad v_i \in V_b, \quad v_j^* \in V_1^*$$

$$\Longrightarrow \quad f_g(v_j^*, v_k) = 0, \quad \forall\, v_k \in V_b, \quad v_k \neq v_i.$$

Proof:

It follows from the previous lemma that $f_g(v_j, v_i) = 1$, $v_i \in V_b$, $v_j \in V_1$ if $f_g(v_i, v_j^*) = 1$. Hence $f_g(v_j, v_k) = 0$, $\forall\, v_k \in V_b$, $v_k \neq v_i$. This result, together with lemma 5, implies $f_g(v_j^*, v_k) = 0$, $\forall\, v_k \in V_b$, $v_k \neq v_i$.

a 8:

Let $f_g(v_i,v_j^*) = 1, \quad v_i \in V_b, \quad v_j^* \in V_1^*.$

$f_g(v_j,V_2^*) = 0.$

f:

If $f_g(v_i,v_j^*) = 1, \quad v_i \in V_b$ and $v_j^* \in V_1^*$, then by a 6

$$f_g(v_i,v_j) = 1$$

contrary to the lemma $f_g(v_j,v_k^*) = 1$, for some $v_k^* \in V_2^*$. further, $f_g(v_k,v_l) = 1, \; v_l \in V_b, \; v_k \neq v_i$. Then the) edges (v_i,v_j^*), (v_j,v_k^*), (v_k,v_l), $e_1, \ldots, e_{k-1}, e_{k+1}, e_{j-1}, e_{j+1}, \ldots, e_p$ will form an independent set ling to a contradiction. //

ıt

+ y //

orem 3:

If $x+y \geq 2$, and $x \neq 0$, then

$$n_{x,y} \leq \frac{p(p-1)}{2} + x(v-2p)+p^2 + \frac{(p-x)(p-x-1)}{2} + y.$$

= z.

of:

It follows from lemma 7 that only $r \leq y$ of the -2p) edges in the set (V_b, V_1^*) will be present in G.

z

Further, if $f_g(v_i,v_j^*) = 1, \quad v_i \in V_b$, and $v_j^* \in V_1^*$, n by lemma 8, $f_g(v_j,V_2^*) = 0$.

Hence by using theorem 2 and the above facts, we get

$$n \leq n_c - n_a - xz - rx - \left\{ y(v-2p) - r \right\}$$

$$= n_c - n_a - y(v-2p) - r(x-1) - zx.$$

Proof:

According to lemma (1)

$$f_g(V_b,V_b) = f_g(V_b,V_0) = f_g(V_b,V_0^*) = 0.$$

If $x = 0$, then $V_2 = \phi$, the null-set. From the definition of V_1, we see that only $y(v-2p)$ edges in the set (V_b, V_1) will be present in G. Further it follows from lemma 7 there are atmost y edges which connect vertices in V_1^* to those in V_b. These results lead to the following:

$$n_{0,y} \leq \frac{2p(2p-1)}{2} + 2y. \qquad //$$

Theorem 6:

$$n_{0,y} \leq n_{0,p} = \frac{2p(2p-1)}{2} + 2p$$

Proof:

It follows from theorem 5 that

$$n_{0,y} \leq n_{0,p} \leq \frac{2p(2p-1)}{2} + 2p$$

The graph shown in Fig.2 has $p_{max} = p$, $x = 0$, $y = p$ and $z = 0$. Since it has $\frac{2p(2p-1)}{2} + 2p$ edges it follows that

$$n_{0,p} \geq \frac{2p(2p-1)}{2} + 2p.$$

Thus

$$n_{0,p} = \frac{2p(2p-1)}{2} + 2p$$

and the graph of Fig.2 is a $G_{0,p}$ graph. It may be seen that this graph consists of a complete subgraph on $(2p+1)$ vertices with all the other vertices isolated. //

<u>Theorem 9</u>:

i) $n_{p,0} \geqslant \alpha$ for $v \geqslant \frac{5p+3}{2}$

ii) $n_{0,p} \geqslant \alpha$ for $v \leqslant \frac{5p+3}{2}$

iii) $n_{p,0} \geqslant n_{1,0}$ iff $v \geqslant \frac{5p}{2}$

iv) $n_{p,0} \geqslant n_{0,p}$ iff $v \geqslant \frac{5p+3}{2}$

v) $n_{1,0} \geqslant n_{0,p}$ iff $v \geqslant 4p$.

vi) $n_{0,p} \geqslant n_{0,0}$

<u>Proof</u>:

Proofs of (i) and (ii) are given in the Appendix. The other results follow after some straightforward manipulations. //

The main results of the paper follow next.

<u>Theorem 10</u>: [8]

a) If $2p \leqslant v \leqslant \frac{5p+3}{2}$ then $G_{0,p}$ is a max-edge v-vertex graph with $p_{max} = p$.

b) If $v \geqslant \frac{5p+3}{2}$, then $G_{p,0}$ is a max-edge v-vertex graph with $p_{max} = p$.

<u>Proof</u>:

It may be easily verified from theorem 9, that for $v \leqslant \frac{5p+3}{2}$, $n_{0,p}$ is greater than α, $n_{1,0}$, $n_{p,0}$ and $n_{0,0}$.

Similarly for $v \geqslant \frac{5p+3}{2}$ $n_{p,0}$ is greater than α, $n_{1,0}$, $n_{0,p}$ and $n_{0,0}$. Hence the result.//

Of the two optimal graphs $G_{p,0}$ and $G_{0,p}$, the latter is connected. In the vulnerability studies of communication nets only connected graphs will be of interest. Hence, in such a case, we may ignore $G_{0,p}$ and look for connected optimal graphs. It is shown in the Appendix that

$$n_{1,0} \geqslant \alpha \quad \text{if} \quad v \leqslant \frac{5p}{2}$$

and

$$n_{p,0} \quad \text{if} \quad v \geqslant \frac{5p}{2}$$

It follows from theorem 9, that $n_{1,0} \geqslant n_{p,0}$ if $v \leqslant \frac{5p}{2}$.

Using these inequalities, we obtain the following main theorem applicable for connected graphs.

<u>Theorem 11</u>:

a) If $v \leqslant \frac{5p}{2}$, $G_{1,0}$ is a max-edge v-vertex connected graph with $p_{max} = p$.

b) If $v \geqslant \frac{5p}{2}$, $G_{p,0}$ is a max-edge v-vertex connected graph with $p_{max} = p$.

III. DESIGN OF v-VERTEX e-EDGE GRAPHS HAVING THE SMALLEST p_{max}

In this section we first obtain a lower bound on p_{max} given the number of vertices and the number of edges. This bound will then be used to design v-vertex e-edge graphs having the smallest p_{max}. Let the functions ρ_i, $i = 1,2,\ldots,5$ be defined as

It follows from theorem 10(b) that

$$\rho_1(p_i,v) \geqslant \rho_i(p_i,v), \quad \because \quad v \geqslant \frac{5p_i + 3}{2}$$

Since $\rho_i(p_i,v) \geqslant e$, this means

$$\rho_1(p_i,v) \geqslant e$$

The above inequality contradicts the definition that p_1 is the smallest value of p satisfying

$$\rho_1(p,v) \geqslant e$$

Hence $p_i \geqslant p_1$ for all $i = 1,2,\ldots, 5.$

b) The proof of part (b) of the theorem follows in a similar manner from theorem 10(a). //

<u>Theorem 13</u>:

a) If $p_1 \leqslant \frac{2v}{5}$ then

$$p_1 = \text{Min}\left\{ p_3, p_4, p_5 \right\}$$

b) If $p_3 \geqslant \frac{2v}{5}$, then

$$p_3 = \text{Min}\left\{ p_1, p_4, p_5 \right\}$$

<u>Proof</u>: The proof follows from theorem 11.//

[illegible] clear from [illegible] theorems 12 [illegible] 13 that

$$[illegible] \quad (12)$$

or

$$[illegible] \quad (13)$$

[illegible]

Remove from $G_{p,0}$ any $(n_{p,0} - e)$ edges making sure that the resulting graph will be connected and has $p_{max} = p$. This can be achieved, for example, by retaining the edges (v_1^*, v_1), (v_1, v_2^*), (v_2^*, v_2), ..., (v_p^*, v_p) and the set (v_1, V_b)

Case-2: If $e > \frac{(4v-1)(2v-3)}{25}$

and $e \leq \frac{8v^2 - 5v}{25}$

and a connected graph is required, repeat case-1.

Case-3: If $e > \frac{(4v-1)(2v-3)}{25}$ and a connected graph is not required then calculate p_2, using (15). Let $p = [p_2]^*$. Construct $G_{0,p}$. Remove from $G_{0,p}$ any $(n_{0,p} - e)$ edges making sure that the resulting graph has $p_{max} = p$.

Case-4: If $e \geq \frac{8v^2 - 5v}{25}$ and a connected graph is required, calculate p_3 using (16) and let $p = [p_3]^*$. Construct $G_{1,0}$ having $p_{max} = p$ and remove from $G_{1,0}$ any $(n_{1,0} - e)$ edges making sure that the resulting graph has $p_{max} = p$ and remains connected.

IV. CONCLUSIONS

In this paper we have considered the following problems:

1. Identification of max-edge v-vertex graphs having a specified edge independence number (Erdos and Gallai theorem [8]).

Substituting $v = \frac{5p+3}{2}$ in the above we get

$$n_{0,p} - \alpha \geqslant \left(\frac{px}{2} - \frac{x^2}{2}\right) + 2p - (2x+y), \text{ for } v \leqslant \frac{5p+3}{2}$$

$$\geqslant 0 \text{ since } p \geqslant x+y$$

Hence the proof.

iv) $$n_{1,0} - \alpha = (v-p) + \frac{2p(2p-1)}{2} - p^2 - \frac{p(p-1)}{2} - \frac{(p-x)(p-x-1)}{2}$$

$$-y - x(v-2p)$$

$$= (x-1)(-v + 2p + \frac{x}{2} + p-x)$$

Substituting $v = \frac{5p}{2}$ in the above we get

$$n_{1,0} - \alpha \geqslant (x-1)(-\frac{p}{2} + \frac{x}{2} + p - x), \quad \text{for} \quad v \leqslant \frac{5p}{2}$$

$$\geqslant (x-1)(-\frac{x}{2} + \frac{p}{2})$$

$$\geqslant 0$$

Hence the proof.