3) *Doubly Exponential Probability Density Function:* The calculation of $\phi^2(\alpha, k_e)$ where

$$k_e(x) = \tfrac{1}{2} \exp(-|x|)$$

is similar to the calculation of $\phi^2(\alpha, k_n)$.

ACKNOWLEDGMENT

The authors wish to thank Dr. H. C. Hamaker and Dr. Mecklenbräucker for their helpful suggestions.

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Bounds on the Sum of Element Sensitivity Magnitudes for Network Functions

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***Abstract*—Bounds on the sum of element sensitivity magnitudes for transfer immittances, transfer voltage, and current ratios are established for networks consisting of resistors, capacitors, inductors, and gyrators. For *RC* and *LC* networks these bounds are given in terms of relevant driving-point functions and their frequency sensitivities. Further, for two-element-kind networks, bounds are given for the quadratic sensitivity index ϕ.**

I. INTRODUCTION

Recently, Smith [1] has obtained, using Vratsano's theorem, bounds on the sum of element sensitivity magnitudes for driving-point impedances (DPI) and for networks consisting of resistors, inductors, capacitors, and gyrators. In this paper we use the adjoint network approach [2] to establish for the same class of networks bounds on the sum of element sensitivity magnitudes for transfer immittances as well as transfer voltage and current ratios.

II. BOUNDS ON THE SUM OF ELEMENT SENSITIVITY MAGNITUDES FOR NETWORK FUNCTIONS

A. Transfer Impedance Z_{12}

Consider a network N consisting of resistors, inductors, capacitors, and gyrators. Let N' denote the adjoint of N. Let Z_{12} be the transfer impedance between ports 1 and 2. Consider an element z_n. It is known from [1] that

$$\frac{\partial Z_{12}}{\partial z_n} = \frac{I_n I_n'}{I_0^2} \tag{1}$$

where $I_n(I_n')$ is the current through the element z_n when port 2(1) of $N(N')$ is excited with a current source of value I_0 and port 1(2) is open circuited. $|I_0|$ refers to the effective value of I_0. It follows from (1) that

$$|S_{z_n}^{Z_{12}}| = \frac{|I_n||I_n'|}{|I_0|^2} \times \frac{|z_n|}{|Z_{12}|} \tag{2}$$

where $S_{z_n}^{Z_{12}}$ is the sensitivity of Z_{12} with respect to z_n. It may be mentioned that the sensitivity of any function F with respect to a parameter p is defined as

$$S_p^F = \frac{p}{F}\frac{\partial F}{\partial p}. \tag{3}$$

If z_n corresponds to a resistor r, then from (1) we get that

$$|S_r^{Z_{12}}| = \frac{|I_r||I_r'|}{|I_0|^2} \times \frac{r}{|Z_{12}|} = \frac{\sqrt{P_r P_r'}}{|I_0|^2 |Z_{12}|} \tag{4}$$

where $P_r(P_r')$ denotes the average power dissipated in the resistor r when port 2(1) of $N(N')$ is excited with a current source of value I_0 and port 1(2) is open circuited.

If z_n represents an inductor l or a capacitor c, then from (2) we get the following:

$$|S_l^{Z_{12}}| = \frac{2\omega\sqrt{W_l W_l'}}{|I_0|^2 |Z_{12}|} \tag{5a}$$

and

$$|S_c^{Z_{12}}| = \frac{2\omega\sqrt{W_c W_c'}}{|I_0|^2 |Z_{12}|} \tag{5b}$$

where $W_l(W_c)$ and $W_l'(W_c')$ denote the average energy stored in the inductor (capacitor) in the network N and N', respectively. It should be noted that N and N' are terminated at the ports as described earlier.

Let $P_R(P_R')$ denote the total average power dissipated in $N(N')$. Then

$$P_R = \sum_r P_r \qquad P_R' = \sum_r P_r'. \tag{6}$$

Let at frequency ω

$$Z_{22} = R_{22} + jX_{22} \qquad Z_{11}' = R_{11}' + jX_{11}'. \tag{7}$$

Using (4), (6), and (7), along with Schwarz's inequality, we get the following:

$$\sum_r |S_r^{Z_{12}}| = \frac{\sum_r \sqrt{P_r P_r'}}{|I_0|^2 |Z_{12}|} \leq \frac{\sqrt{P_R P_R'}}{|I_0|^2 |Z_{12}|} = \frac{\sqrt{R_{22} R_{11}'}}{|Z_{12}|} = \frac{\sqrt{R_{22} R_{11}}}{|Z_{12}|}. \tag{8}$$

The last step in (8) follows from the relation $Z_{11}' = Z_{11}$ [3].

It may be noted that in the case of a DPI, (8) will reduce to [1, eq. 45]. Further, if N consists of r, l, and c elements only, then for a DPI, (8) will reduce to [1, eq. 11] since the adjoint of N is itself if N is reciprocal.

Let

$$W_C = \sum_c W_c \qquad W_C' = \sum_c W_c' \qquad W_L = \sum_l W_l$$
$$W_L' = \sum_l W_l' \qquad W = W_C + W_L \qquad W' = W_C' + W_L'. \tag{9}$$

Using (5), (7), and (9), along with Schwarz's inequality we get the following:

$$\sum_l |S_l^{Z_{12}}| + \sum_c |S_c^{Z_{12}}| = \frac{2\omega\left(\sum_l \sqrt{W_l W_l'} + \sum_c \sqrt{W_c W_c'}\right)}{|I_0|^2 |Z_{12}|} \leq \frac{2\omega\sqrt{WW'}}{|I_0|^2 |Z_{12}|}. \tag{10}$$

If the network N is reciprocal, then using (5), [1, eq. 12] for a DPI can be easily established in a similar manner.

It is known [4] that for the network under consideration

$$\sum_l S_l^F + \sum_c S_c^F = S_\omega^F \tag{11}$$

where F is any network function. Hence we get

$$\sum_l |S_l^F| + \sum_c |S_c^F| \geq |S_\omega^F|. \tag{12}$$

Manuscript received October 26, 1971. This work was supported by the National Research Council of Canada under Grant A-7739. This work is part of a dissertation to be submitted by C. Bhushan, under the supervision of M. N. S. Swamy, to Sir George Williams University, Montreal, Que., Canada.
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From (10) and (12) we get the following:

$$|S_\omega^F| \le \sum_l |S_l^{Z_{12}}| + \sum_c |S_c^{Z_{12}}| \le \frac{2\omega\sqrt{WW'}}{|I_0|^2|Z_{12}|}. \tag{13}$$

If the network consists of r and c elements only, then it is known that [4], [5]

$$\sum_r S_r^{Z_{12}} = 1 + S_\omega^{Z_{12}} \qquad \sum_c S_c^{Z_{12}} = S_\omega^{Z_{12}}. \tag{14}$$

Hence, on taking magnitudes we have

$$\sum_r |S_r^{Z_{12}}| + \sum_c |S_c^{Z_{12}}| \ge |1 + 2S_\omega^{Z_{12}}|. \tag{15}$$

Further, for an RC network we can show starting from (5b) that

$$\sum_c |S_c^{Z_{12}}| \le (\sqrt{|X_{11}X_{22}|})/|Z_{12}|. \tag{16}$$

Then from (8), (15), and (16) we obtain the following inequality:

$$|1 + 2S_\omega^{Z_{12}}| \le \sum_r |S_r^{Z_{12}}| + \sum_c |S_c^{Z_{12}}| \le \frac{\sqrt{R_{11}R_{22}} + \sqrt{|X_{11}X_{22}|}}{|Z_{12}|}. \tag{17}$$

An inequality similar to (17) can be obtained easily for an RL network.

Let the network N consist of only inductors and capacitors. It is known that [6]

$$W = W_C + W_L = \frac{1}{2}|I_0|^2 \frac{\partial X_{22}}{\partial \omega}$$

and

$$W' = W_L' + W_c' = \frac{1}{2}|I_0|^2 \frac{\partial X_{11}}{\partial \omega}. \tag{18}$$

Hence from (10) and (18) we get the following inequality:

$$\sum_l |S_l^{Z_{12}}| + \sum_c |S_c^{Z_{12}}| \le \frac{\omega}{|Z_{12}|}\sqrt{\frac{\partial X_{22}}{\partial \omega}\frac{\partial X_{11}}{\partial \omega}} = \sqrt{(|S_\omega^{Z_{22}}|\,|S_\omega^{Z_{11}}|\,|Z_{11}|\,|Z_{22}|)}/|Z_{12}|. \tag{19}$$

Thus from (13) and (19) for an LC network we get the following inequality:

$$|S_\omega^{Z_{12}}| \le \sum_l |S_l^{Z_{12}}| + \sum_c |S_c^{Z_{12}}| \le \sqrt{(|S_\omega^{Z_{22}}|\,|S_\omega^{Z_{11}}| \times |Z_{11}|\,|Z_{22}|)}/|Z_{12}|. \tag{20}$$

This completes our discussion of the bounds on the sum of element sensitivity magnitudes for transfer impedance functions.

B. Transfer Admittance Y_{12}

Let Y_{12} denote the transfer admittance between ports 1 and 2 of a network N. Let port 2(1) of $N(N')$ be excited with a voltage source of value V_0 and port 1(2) be short circuited. Let

$$Y_{22} = G_{22} + jB_{22} \qquad Y_{11}' = G_{11}' + jB_{11}'$$

where $Y_{22}(Y_{11}')$ is the short-circuit driving-point admittance across port 2(1) of $N(N')$. The quantities W_l, W_c, W_l', W_c', etc. are the same as in the case of transfer impedances, except that N and N' are now terminated as mentioned above.

With these definitions all inequalities and equations from (4) to (20) hold true if Z_{12}, I_0, R_{11}, R_{22}, X_{11}, and X_{22} are replaced, respectively, by Y_{12}, V_0, G_{11}, G_{22}, B_{11}, and B_{22}, and in (14), (15), and (17) the quantity $(2S_\omega^{Z_{12}}+1)$ is replaced by $(2S_\omega^{Y_{12}}-1)$.

C. Transfer Voltage Ratio T_V

Let T_V be the transfer voltage ratio between ports 2 and 1 of a network N. Let port 1(2) of $N(N')$ be excited with a voltage (current) source of value $V_1(I_2)$ and port 2(1) of $N(N')$ be open circuited (short circuited). Let $W(W')$ denote the average energy stored in the reactive elements of $N(N')$ when it is terminated as above. Let for N and N'

$$\boldsymbol{Y}_{11} = 1/Z_{11} = \boldsymbol{G}_{11} + j\boldsymbol{B}_{11} \quad \text{and} \quad \boldsymbol{Z}_{22}' = 1/Y_{22}' = \boldsymbol{R}_{22}' + j\boldsymbol{X}_{22}'.$$

Following the same approach as used in the case of Z_{12}, we can establish bounds on the sum of element sensitivity magnitudes for T_V. These results are summarized in Table I.

TABLE I

BOUNDS ON THE SUM OF ELEMENT SENSITIVITY MAGNITUDES FOR T_V

Network Type	Bounds
R, L, C, and Gyrators	$\sum_r \lvert S_r^{T_V}\rvert \le (\sqrt{\boldsymbol{G}_{11}\boldsymbol{R}_{22}}/\lvert T_V\rvert)$
	$\lvert S_\omega^{T_V}\rvert \le \sum_l \lvert S_l^{T_V}\rvert + \sum_c \lvert S_c^{T_V}\rvert$
	$\le (2\omega\sqrt{WW'})/\lvert V_1\rvert\lvert I_2\rvert\,\lvert T_V\rvert$
R and C	$\lvert S_\omega^{T_V}\rvert \le \sum_r \lvert S_r^{T_V}\rvert \le \sqrt{\boldsymbol{G}_{11}\boldsymbol{R}_{22}}/\lvert T_V\rvert$
	$\lvert S_\omega^{T_V}\rvert \le \sum_c \lvert S_c^{T_V}\rvert \le \sqrt{\lvert\boldsymbol{B}_{11}\boldsymbol{X}_{22}\rvert}/\lvert T_V\rvert$
L and C	$\lvert S_\omega^{T_V}\rvert \le \sum_l \lvert S_l^{T_V}\rvert + \sum_c \lvert S_c^{T_V}\rvert$
	$\le (1/\lvert T_V\rvert)\sqrt{\lvert S_\omega^{\boldsymbol{Y}_{11}}\rvert\,\lvert S_\omega^{\boldsymbol{Z}_{22}}\rvert\,\lvert \boldsymbol{Y}_{11}\rvert\,\lvert \boldsymbol{Z}_{22}}$

D. Transfer Current Ratio T_c

Let T_c denote the transfer current ratio between ports 2 and 1 of a network N. Let $W(W')$ denote the total average energy stored in the reactive elements of $N(N')$ when port 1(2) of $N(N')$ is excited with a current (voltage) source of value $I_1(V_2)$ and port 2(1) of $N(N')$ is short circuited (open circuited). Let for N and N'

$$\boldsymbol{Z}_{11} = 1/Y_{11} = \boldsymbol{R}_{11} + j\boldsymbol{X}_{11} \quad \text{and} \quad \boldsymbol{Y}_{22}' = 1/Z_{22}' = \boldsymbol{G}_{22}' + j\boldsymbol{B}_{22}'.$$

Observing that the transfer current ratio between ports 2 and 1 of N is the same as the transfer voltage ratio between ports 1 and 2 of N', it can be shown that all the inequalities of Table I will hold true if I_2, V_1, $\boldsymbol{G}_{11}$, $\boldsymbol{B}_{11}$, $\boldsymbol{R}_{22}$, and $\boldsymbol{X}_{22}$ are replaced, respectively, by I_1, V_2, $\boldsymbol{G}_{22}$, $\boldsymbol{B}_{22}$, $\boldsymbol{R}_{11}$, and $\boldsymbol{X}_{11}$.

III. BOUNDS ON QUADRATIC SENSITIVITY INDEX

A commonly used criterion of performance is the quadratic sensitivity index ϕ given by [7]

$$\phi = \sum_{x_i} |S_{x_i}^F|^2.$$

In the previous discussion we have obtained for two-element-kind networks bounds for the sum of magnitude sensitivities with respect to all the elements of the network. Using these results we can obtain bounds for the function ϕ. For an RC network, we get from (8) and (16) the following:

$$\sum_r |S_r^{Z_{12}}|^2 \le R_{11}R_{22}/|Z_{12}|^2 \tag{21a}$$

$$\sum_c |S_c^{Z_{12}}|^2 \le X_{11}X_{22}/|Z_{12}|^2. \tag{21b}$$

Hence

$$\phi = \sum_r |S_r^{Z_{12}}|^2 + \sum_c |S_c^{Z_{12}}|^2 \le \frac{R_{11}R_{22}}{|Z_{12}|^2} + \frac{X_{11}X_{22}}{|Z_{12}|^2}. \tag{22}$$

Using (14) and the Chebyshev inequality

$$\left(\sum_{i=1}^n a_i\right)^2/n \le \sum_{i=1}^n a_i^2 \tag{23}$$

we can show that

$$\phi \ge \frac{1}{n_1}|1 + S_\omega^{Z_{12}}|^2 + \frac{1}{n_2}|S_\omega^{Z_{12}}|^2 \tag{24}$$

TABLE II

BOUNDS ON QUADRATIC SENSITIVITY INDEX ϕ

F	RC Network[a]	LC Network[b]
Z_{ij}	$\frac{1}{n_1}\lvert 1+S_\omega^F\rvert^2+\frac{1}{n_2}\lvert S_\omega^F\rvert^2 \le \phi \le (R_{11}R_{22}+\lvert X_{11}\rvert\lvert X_{22}\rvert)/\lvert F\rvert^2$	$\frac{1}{4n_1}\lvert S_\omega^F-1\rvert^2+\frac{1}{4n_2}\lvert S^F{}_\omega+1\rvert^2 \le \phi \le (\lvert S_\omega^{Z_{22}}\rvert\lvert S_\omega^{Z_{11}}\rvert\lvert Z_{11}\rvert\lvert Z_{22})/\lvert F\rvert^2$
Y_{ij}	$\frac{1}{n_1}\lvert -1+S_\omega^F\rvert^2+\frac{1}{n_2}\lvert S_\omega^F\rvert^2 \le \phi \le (G_{11}G_{22}+\lvert B_{11}\rvert\lvert B_{22}\rvert)/\lvert F\rvert^2$	$\frac{1}{4n_1}\lvert S_\omega^F+1\rvert^2+\frac{1}{4n_2}\lvert S_\omega^F-1\rvert^2 \le \phi \le (\lvert S_\omega^{Y_{22}}\rvert\lvert S_\omega^{Y_{11}}\rvert\lvert Y_{11}\rvert\lvert Y_{22}\rvert)/\lvert F\rvert^2$
T_V	$\left(\frac{1}{n_1}+\frac{1}{n_2}\right)\lvert S_\omega^F\rvert^2 \le \phi \le (G_{11}R_{22}+\lvert B_{11}\rvert\lvert X_{22}\rvert)/\lvert F\rvert^2$	$\frac{1}{4}\left(\frac{1}{n_1}+\frac{1}{n_2}\right)\lvert S_\omega^F\rvert^2 \le \phi \le (\lvert S_\omega^{Y_{11}}\rvert\lvert S_\omega^{Z_{22}}\rvert\lvert Y_{11}\rvert\lvert Z_{22}\rvert)/\lvert F\rvert^2$
T_c	$\left(\frac{1}{n_1}+\frac{1}{n_2}\right)\lvert S_\omega^{F2}\rvert \le \phi \le (G_{22}R_{11}+\lvert B_{22}\rvert\lvert X_{11}\rvert)/\lvert F\rvert^2$	$\frac{1}{4}\left(\frac{1}{n_1}+\frac{1}{n_2}\right)\lvert S_\omega^F\rvert^2 \le \phi \le (\lvert S_\omega^{Y_{22}}\rvert\lvert S_\omega^{Z_{11}}\rvert\lvert Y_{22}\rvert\lvert Z_{11}\rvert)/\lvert F\rvert^2$

[a] n_1 = number of resistors; n_2 = number of capacitors.
[b] n_1 = number of capacitors; n_2 = number of inductors.

where n_1 is the number of resistors and n_2 is the number of capacitors in the network.

Similar bounds can be obtained for other network functions of two-element-kind networks. These are summarized in Table II.

IV. CONCLUSIONS

For networks containing R, L, and C elements and gyrators, bounds have been obtained for the sum of magnitude of sensitivities of network functions with respect to R, C, and L. In the case of two-element-kind networks these bounds have been expressed in terms of frequency sensitivities and the real and imaginary parts of relevant driving-point functions. Further, bounds have been obtained for the quadratic sensitivity index ϕ for the case of two-element-kind networks.

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Simple Bounds on the Variance of System Functions of Two-Element-Kind Networks Having Randomly Chosen Elements

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***Abstract*—The elements of the networks under consideration are assumed to have small statistically independent random variations about their nominal values. With these assumptions, it is shown that the variance of a given system function can be related to its sensitivity with respect to the network elements. Recent results in sensitivity analysis are then used to establish simple upper bounds on the variance of two-port network functions.**

Manuscript received December 6, 1971; revised February 18, 1972.
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INTRODUCTION

Let F be a system function of an arbitrary network composed of resistors, capacitors, and inductors having nominal values r_i, c_i, and l_i. Let Δx_i be a small random fluctuation of a given element about its nominal value x_i such that $E[\Delta x_i]=0$. Then the fluctuation ΔF_{x_i} of the system function about its nominal value may be approximated by

$$\Delta F_{x_i} \doteq \frac{\partial F}{\partial x_i}\Delta x_i.$$

Now defining the normalized fluctuation of the element as $\epsilon_i=\Delta x_i/x_i$, the normalized fluctuation $V_{x_i}{}^F \triangleq \Delta F_{x_i}/F$ of the system function F may be expressed as

$$V_{x_i}{}^F = \frac{\partial F}{\partial x_i}\frac{x_i}{F}\epsilon_i = S_{x_i}{}^F\epsilon_i.$$

For the whole set of elements,[1] we have

$$V_x{}^F = \sum_i S_{x_i}{}^F\epsilon_i, \qquad x = \{x_1, x_2, \cdots, x_\kappa\}. \tag{3}$$

It is evident that $E[V_x{}^F]=0$. Assuming that the random variables ϵ_i are statistically independent, we have that $E[\epsilon_i\epsilon_j]=0$; $i\neq j$, and hence [1]

$$E\lvert V_x{}^F\rvert^2 = \sum_i \lvert S_{x_i}(j\omega)\rvert^2\sigma_{\epsilon_i}{}^2, \qquad \sigma_{\epsilon_i}{}^2 = E[\epsilon_i{}^2]. \tag{4}$$

We now note that all those elements which have been chosen according to the same probability law (e.g., uniformly distributed 10-percent resistors) will have identical normalized variances. In such cases we will write $\sigma_{\epsilon_i}{}^2=\sigma_x{}^2$ where now x denotes the class of elements defined by

$$x = \{x_i : E[\Delta x_i/x_i]^2 = \sigma_x{}^2\}.$$

Equation (4) now becomes

$$E\lvert V_x{}^F\rvert^2 = \sigma_x{}^2\sum_i \lvert S_{x_i}{}^F\rvert^2. \tag{5}$$

In what follows we will assume that the sets of resistors, capacitors, and inductors each constitute such a class.[2]

Bounds on the Variance of $V_x{}^F$

We will now make use of recent results in sensitivity analysis [2], [3], to formulate upper bounds on $E\lvert V_x{}^F\rvert^2$.

Driving Point Functions: Smith [2] has shown for driving point functions composed of resistors, capacitors, inductors, and ideal transformers that

$$\sum_i \lvert S_{r_i}{}^{Z_0}\rvert = \frac{R_0}{\lvert Z_0\rvert}, \qquad Z_0 = R_0 + jX_0 \tag{6a}$$

where $S_{r_i}{}^{Z_0}$ is the sensitivity of the impedance Z_0 with respect to the

[1] The inequalities obtained will still be valid for the more general case if we let $\sigma_x{}^2=\max_i\{\sigma_{\epsilon_i}{}^2\}$.
[2] A similar measure has been formulated by Rosenblum and Ghausi [4].