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CONTINUOUSLY EQUIVALENT REALIZATIONS OF 3RD-ORDER PARAMOUNT MATRICES

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SUMMARY

The continuously equivalent realizability of 3rd-order real symmetric paramount matrices is discussed.

1. INTRODUCTION

A necessary condition for an $(n \times n)$ real symmetric matrix to be realizable as the shortcircuit conductance or open-circuit resistance matrix of a resistive n-port network is that the matrix be paramount. This condition is not sufficient, in general. However, Tellegen[1] demonstrated the sufficiency of this condition by giving a general network configuration capable of realizing any 3rd-order paramount matrix. Weinberg[2] has reported the details of Tellegen's procedure. Recently, a new procedure based on an alternative network configuration was presented.[3] Tellegen's procedure, as well as the one given in Reference 3, leads to a unique realization for a given (3×3) paramount matrix.

In this paper, we consider the question of the continuously equivalent realizability of 3rd-order paramount matrices.

A (3×3) symmetric matrix with positive diagonal entries can have any one of the following two distinct patterns:

Pattern 1:

y_{12} and y_{23} positive
y_{13} negative

Pattern 2:

y_{12} positive
y_{12} and y_{23} negative

The results of this paper are based on the following theorems, proofs of which are avilable in Reference 4.

Theorem 1

In a (3×3) real symmetric paramount matrix $\mathbf{Y}$, whose off diagonal entries have the sign pattern 1, at least two rows are dominant.

Theorem 2

In a 3rd-order real symmetric paramount matrix $\mathbf{Y}$ having the sign pattern 1, $y_{11} \neq y_{12}$.

Theorem 3

Let a real symmetric paramount matrix satisfy the following conditions:

(i) It has sign pattern 2,
(ii) $|y_{13}| \neq |y_{23}|$, $|y_{23}| \leq \min\{y_{12}, |y_{13}|\}$.

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Then

$$y_{11} \neq y_{12}$$

Theorem 4

Let a real symmetric (3×3) paramount matrix **y** satisfy the following conditions: y_{13} is negative, y_{12} is positive and $y_{23}=0$. Then
(a) rows 2 and 3 are dominant,
(b) if one of these rows is marginally dominant, then row 1 is also dominant.

Theorem 5

Let a real symmetric (3×3) paramount matrix satisfy the following conditions:
(i) It has sign pattern 2,
(ii) $y_{22}=y_{12}$,
(iii) $|y_{23}| \leq \min\{y_{12}, |y_{13}|\}$.
Then the matrix **Y** can be bought to the uniformly tapered form.

Theorem 6

If, in a real symmetric (3×3) paramount matrix **Y**,
(i) y_{12}, is positive, y_{13} and y_{23} are negative and are equal,
(ii) $|y_{13}|=|y_{23}|\leq y_{12}$.
Then the matrix **Y** can be brought to the uniformly tapered form. Theorem 7 follows from the results of Reference 5.

Theorem 7[5]

Let, in a real symmetric (3×3) matrix **Y**,

$$y_{22}=|y_{12}| \qquad y_{33}=|y_{13}|$$

$$|y_{12}|>0 \qquad |y_{13}|>0 \qquad |y_{23}|>0$$

If **Y** is realizable as the **Y** matrix of a 3-port network containing more than four terminals, then one of the following is true:
(i) $y_{22}=y_{33}=|y_{23}|$
(ii) $y_{22}=|y_{23}|$
(iii) $y_{33}=|y_{23}|$.
The implication of the above theorem is that, if, in a (3×3) real symmetric matrix, $y_{22}=|y_{12}|$ and $y_{33}=|y_{13}|$, and, if all the off diagonal entries are distinct and nonzero, then the matrix cannot be realized by a 3-port network having more than 4 terminals.

2. CONTINUOUSLY EQUIVALENT REALIZATIONS OF 3RD-ORDER REAL SYMMETRIC PARAMOUNT MATRICES HAVING NO ZERO ENTRIES

We give, in Section 2, a network configuration for realizing any 3rd-order real symmetric paramount matrix **Y**, having no zero off diagonal entries, as a shortcircuit conductance matrix. This configuration enables us to generate continuously equivalent realizations for a given matrix. Let the configuration of the 3-ports of the network be as shown in Figure 1. The realization of the matrix **Y** having sign pattern 1 is discussed under case 1, and that of the matrix having sign pattern 2 under case 2.

Case *1*

Let the given **Y** matrix satisfy the following conditions:
(i) The offdiagonal entries have the sign pattern 1,
(ii) rows 2 and 3 are dominant,
(iii) $|y_{13}|y_{22}(y_{33}-y_{23}-y_{12})\geqslant 0$.
It can be shown that conditions (ii) and (iii) involve no loss of generality.

The edge conductances of the network of departure[6] N_d having **Y** as its shortcircuit conductance matrix and having the port configuration of Figure 1 are given by the following equations:

$$(g_{12})_d=(y_{11}-y_{12}), \qquad (g_{13})_d=y_{12} \qquad (g_{14})_d=y_{13}$$
$$(g_{15})_d=-|y_{13}| \qquad (g_{23})_d=(y_{22}-y_{12})$$
$$(g_{24})_d=-(|y_{13}|+y_{23}) \qquad (g_{25})_d=(|y_{13}|+y_{23})$$
$$(g_{34})_d=y_{23} \qquad (g_{35})_d=-y_{23}$$

and

$$(g_{45})_d=y_{33} \tag{1}$$

The edge conductances of a suitable padding 3-port network are to be obtzined from the following expressions:

$$(g_{ij})_p=-s_is_j \qquad i,j=1,2,3 \qquad i\neq j \qquad \text{or } i,j=4,5 \qquad i\neq j$$
$$(g_{ij})_p=s_is_j \qquad i=1,2,3 \quad \text{and} \quad j=4,5 \tag{2}$$

where

$$s_1+s_2+s_3=s_4+s_5$$

We have to choose suitable nonnegative values for s_i's such that

$$(g_{ij})=(g_{ij})_d+(g_{ij})_p\geqslant 0$$

for all i and $j=1, 2, 3, 4, 5$, $i\neq j$. Let us choose s_i's such that g_{13}, g_{15} and g_{23} are zero. Then we can express s_1, s_2, s_5 and s_4 in terms of s_3 as follows:

$$s_1=y_{12}/s_3 \qquad s_5=(|y_{13}|s_3)/y_{12}$$
$$s_2=(y_{22}-y_{12})/s_3$$
$$s_4=\frac{1}{s_3}\,\frac{s_3^2(y_{12}-|y_{13}|)+y_{12}y_{22}}{y_{12}} \tag{3}$$

It can be seen from (1) and (2) that g_{13}, g_{25} and g_{34} are nonnegative. The requirement that the remaining conductances be nonnegative helps to establish the limits within which s_3 should lie. These limits can be obtained as follows:

Case 1(a)

$$|y_{13}|>y_{12}$$

and

$$y_{12}y_{33}>|y_{13}|y_{22}$$
$$\max\{C_1, C_3\}\leqslant s_3^2\leqslant C_2$$

Case 1(b)

$$|y_{13}|>y_{12}$$

and

$$|y_{13}|y_{22} > y_{12}y_{33}$$
$$\max\{C_1, C_3, C_4\} \leqslant s_3^2 \leqslant C_2$$

Case 1(c)

$$y_{12} > |y_{13}|$$

and

$$y_{12}y_{33} > |y_{13}|y_{22}$$
$$\max\{C_1, C_3\} \leqslant s_3^2 \leqslant C_2$$

Case 1(d)

$$y_{12} > |y_{13}|$$

and

$$|y_{13}|y_{22} > y_{12}y_{33}$$
$$\max\{C_1, C_3\} \leqslant s_3^2 \leqslant C_2$$

where

$$C_1 = y_{12}(y_{22} - y_{12})/(y_{11} - y_{12})$$
$$C_2 = \frac{y_{12}y_{22}(y_{22} - y_{12})}{y_{12}(|y_{13}| + y_{23}) - (y_{22} - y_{12})(y_{12} - |y_{13}|)}$$
$$C_3 = y_{12}y_{23}/|y_{13}|$$
$$C_4 = \frac{y_{12}(|y_{13}|y_{22} - y_{12}y_{33})}{(|y_{13}| - y_{12})|y_{13}|} \quad (4)$$

It must be pointed out that, in case 1d above, C_1, C_2 and C_3 should be calculated using the matrix obtained by first interchanging the rows and columns 2 and 3 of the given matrix **Y**, and then multiplying the entries of row 1 and column 1 of the resulting matrix by -1.

Once a choice of s_3^2 is made satisfying the constraints given above, s_1, s_2, s_4 and s_5 can be calculated using (3), and the nonzero conductances of the required network can be calculated using (1) and (2).

While the above approach leads to continuously equivalent networks containing seven conductances, for limiting values of the parameter s_3^2, we obtain six conductance realizations also.

Case 2

Let the given paramount matrix **Y** satisfy the following constraints:
(i) The offdiagonal entries have the sign pattern 2,
(ii) $|y_{23}| \leqslant \min\{y_{12}, |y_{13}|\}$, $y_{13} \neq y_{23}$ and $y_{22} \neq y_{12}$,
(iii) $|y_{13}|y_{22}(y_{33} - y_{23} - |y_{13}|) \geqslant y_{12}y_{33}(y_{22} - y_{23} - y_{12}) \geqslant 0$.
Proceeding as in case 1, we obtain the following limits on s_3:

Case 2a

$|y_{13}| > y_{12}$ and $y_{12}y_{33} > |y_{13}|y_{22}$, $C_1 \leqslant s_3^2 \leqslant C_2$.

Case 2b

$|y_{13}| > y_{12}$ and $|y_{13}|y_{22} > y_{12}y_{33}$, $\max\{C_1, C_4\} \leqslant s_3^2 \leqslant C_2$.

Case 2c

$y_{12} > |y_{13}|$ and $y_{12}y_{33} > |y_{13}|y_{22}$, $C_1 \leqslant s_3^2 \leqslant C_2$.

Case 2d

$y_{12} > |y_{13}|$ and $|y_{13}|y_{22} > y_{12}y_{33}$, $C_1 \leq s_3^2 \leq C_2$.

The discussions following (4) are applicable here too.

3. REALIZATION OF DEGENERATE PARAMOUNT MATRICES

The synthesis procedures of Section 2 are applicable to the class of matrices having no zero entries. Realization of matrices having two zero entries can be achieved by inspection, and the networks so realized will be in two connected parts.

If y_{12} or y_{13} is equal to zero, then the approach adopted in Section 2 requires, as can be seen from (1) and (2), that one more y_{ij} be equal to zero. On the other hand, if $y_{23} = 0$, then the techniques of Section 2 will be applicable, except when $y_{22} = y_{12}$. Under case 3, we give a synthesis procedure for matrices in which only one of the y_{ij}s is equal to zero.

In case 2, it was assumed that $y_{13} \neq y_{23}$ and $y_{22} \neq y_{12}$. Realization of matrices in which such conditions are not satisfied will be dealt with under case 4.

Case 3

Let the matrix **Y** satisfy the following conditions:

(i) $y_{23} = 0$,
(ii) $y_{22} = y_{12}$,
(iii) y_{13} is negative.

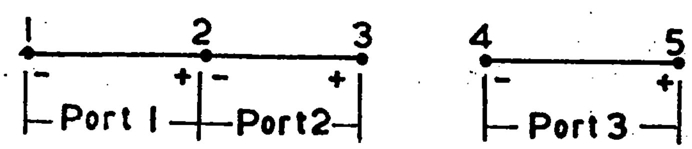

Figure 1. Port configuration for cases 1, 2 and 3

For this case, we assume the port configuration of Figure 1. Once again, we first obtain the conductances of the network of departure N_d, and choose suitable nonnegative s_i's so that the final conductances of the required network will be nonnegative. For this purpose, we choose s_i's as follows:

$$s_1 = (y_{11} - y_{12})/s_{12}$$

$$s_3 = 0 \qquad s_4 = \frac{|y_{13}|}{s_2} \qquad s_5 = \frac{1}{s_2}[y_{11} - y_{12} - |y_{13}| + s_2^2]$$

where s_2 has to be chosen such that

$$s_2 \geq (y_{11} - y_{12} - |y_{13}|)/(y_{33} - |y_{13}|)$$

In the final network obtained, we find

$$g_{24} = 0 \qquad g_{23} = g_{34} = g_{35} = g_{12} = 0$$

The above approach does not pose any problem, except when row 3 is also marginally dominant. This case is discussed in Reference 4.

Case 4

Let the matrix **Y** satisfy the following conditions:

(i) The offdiagonal entries have the sign pattern 2,
(ii) $y_{22} = y_{12}$ or $y_{13} = y_{23}$ or $y_{22} = y_{12}$ and $y_{23} = y_{13}$,
(iii) $y_{23} \neq 0$, $y_{33} \neq |y_{13}|$,
(iv) $|y_{23}| \leq \min\{y_{12}, |y_{13}|\}$.

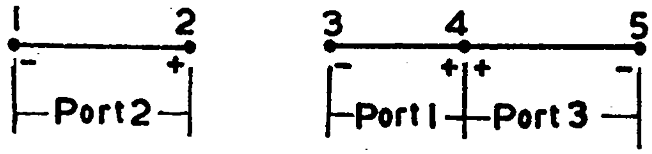

Figure 2. Port configuration for case 4

We assume, for this case, the port configuration of Figure 2. We choose $g_{13}=g_{24}=g_{25}=0$. s_1, s_2, s_4 and s_5 are obtained as

$$s_1=\frac{y_{12}}{s_3} \qquad s_4=\frac{(y_{12}-|y_{23}|)}{s_2}$$

$$s_5=\frac{|y_{23}|}{s_2} \qquad s_3=s_2$$

where

$$s_2^2 \geq (y_{12}-|y_{23}|)|(y_{23})|/(y_{33}-|y_{13}|)$$

The above approach is applicable even when $y_{33}=|y_{13}|$, provided that $y_{12}=|y_{23}|$.For, in such a case, any arbitrary value may be chosen for s_2^2. If, instead, $y_{13}=y_{23}$, the procedure can still be applied after interchanging the rows and columns 2 and 3 of the given matrix **Y** and multiplying the entries of row 1 and column 1 of the resulting matrix by -1. Thus the approach of case 4 is also applicable to all the (3×3) paramount matrices satisfying the following conditions:

(i) The offdiagonal entries have the sign pattern 2,

(ii) $y_{22}=y_{12}$, $y_{33}=|y_{13}|$,

(iii) $|y_{23}|=y_{12}\leq|y_{13}|$ or $|y_{23}|=|y_{13}|\leq y_{12}$.

The only case that is yet to be considered is that of a matrix in which y_{13} and y_{23} are nonnegatives, $|y_{13}|$, $|y_{23}|$ and $|y_{12}|$ are distinct and nonzero and $y_{33}=|y_{13}|$ and $y_{22}=y_{12}$.

By Theorem 7, for such a matrix, a 5-node realization does not exist, and hence continuously equivalent realizations are not possible. However, it can be realized by a 4-node network after converting it to the uniformly tapered form.

4. CONCLUSIONS

It has been shown, in this paper, that continuously equivalent realizations are possible in the case of all 3rd-order paramount **Y** matrices except those in which $y_{22}=y_{12}$, $y_{33}=|y_{13}|$, $|y_{13}|$, $|y_{23}|$ and $|y_{12}|$ are distinct and nonzero. However for these matrices, no 5-node realization exists, and hence no continuously equivalent realizations are possible.

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