Minimum Order Graphs with Specified Diameter, Connectivity, and Regularity

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Relationships among graph invariants such as the number of vertices, diameter, connectivity, maximum and minimum degrees, and regularity are being studied recently, motivated by their usefulness in the design of fault-tolerant and low-cost communication and interconnection networks. A graph is called a (d,c,r) graph if it has diameter d, connectivity c, and regularity r. The minimum number of vertices in $(d,1,3)$, $(d,2,3)$, $(d,3,3)$, and (d,c,c) graphs have been reported in the literature. In this paper, the minimum number of vertices in a (d,c,r) graph with $r > c$ is determined, thereby exhausting all the possible choices of values for d, c, and r. Our proof is constructive and hence we get a collection of optimal (d,c,r) graphs.

1. INTRODUCTION

The ever-increasing complexity of communication and interconnection networks has provided the motivation for several studies on fault-tolerant and low-cost network design problems. Graph invariants such as the number of vertices, number of edges, diameter, connectivity, regularity, maximum and minimum degrees directly contribute to the cost or fault-tolerance of a network, and the interrelations between these parameters can be profitably used in algorithms determining these parameters for a given graph or in the algorithms for designing optimal networks with prescribed topological properties. Some of the recent studies in this area are [1–7, 9–17].

Klee and Quaife [12] considered the problem of maximizing the distance over which communication could be achieved in a network. The network should be capable of tolerating at least $c - 1$ failures of the communication stations and the resource used should be minimum, the resource being the number of stations. Also each station should be able to communicate directly with at least r stations. If the maximum distance

This work was supported by the Natural Sciences and Engineering Research Council of Canada under Grants A-7739 and A-4680.

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NETWORKS, Vol. 19 (1989) 25–46

CCC 0028-3045/89/010025-22$04.00

between any two stations which communicate directly can be taken as one unit, then the following graph-theoretic problems arise:

1. Given d, c, and r, construct networks with minimum number of vertices, having diameter $\geqslant d$, connectivity $\geqslant c$, and degree of every vertex $\geqslant r$.

2. Given n, c, and r, construct networks with maximum diameter, having the number of vertices $\leqslant n$, connectivity $\geqslant c$, and degree of every vertex $\geqslant r$.

In [12], the first problem was solved. The problems become difficult when the inequalities are replaced by equalities. In [11,13] the minimum number of vertices for an r-regular graph with diameter d and connectivity $\geqslant c$ are determined for the combinations $(d,1,3)$, $(d,2,3)$, and $(d,3,3)$ (for odd d in the last case) of (d,c,r), and such graphs are classified and enumerated. It is also mentioned in [11–13] that they have determined the minimum number of vertices for all the combinations of (d,c,r) such that d is the diameter, connectivity $\geqslant c$, and r is the regularity.

Alternative approaches for the cases $(d,1,3)$, $(d,3,3)$, and $(d,1,r)$ have been given by Myers [14–16]. Recently, Bhattacharya [1] has considered the case (d,c,c).

While Klee and Quaife consider graphs with connectivity (the minimum number of vertices required to disconnect the graph or make it trivial) $\geqslant c$, the others consider the case where connectivity is equal to c. Our results show that the difference between these two approaches surface only when $d = 2$.

In this paper, $\mu(d,c,r)$, defined as the minimum number of vertices in a graph of diameter d, connectivity c, and regularity r is determined for $d \geqslant 2$, $c \geqslant 1$, and $r > c$, thus exhausting all the cases. The proof is constructive in nature and many cases are to be considered. Though the methods of construction differ slightly in different cases, it is interesting to note that the formulas for μ coincide in many cases. Also the same formulas hold good for the case $r = c$ considered in [1].

In [7,10], the lower bounds for the connectivity of a graph or a digraph have been given in terms of the number of vertices, diameter, and maximum degree, and, it is mentioned in [7] that these are found to be useful in the construction of large graphs with given diameter and maximum degree. This construction of the (Δ,D) graphs is receiving considerable attention in the current literature because of its usefulness in the design of communication networks. Our result on μ directly provides an upper bound to the connectivity of an r-regular graph of diameter d on n vertices; it also provides an upper bound for the diameter, given the other parameters n, c, and r.

Boesch and Wang considered an important class of graphs called circulants and determined an upper bound for their diameters in [6]. They also mention that the connectivity of a circulant can be easily determined. Since the circulants are regular graphs, an upper bound for the diameter of a circulant can be readily determined now in terms of the other parameters.

Throughout this paper only undirected simple graphs without loops or multiple edges are considered. Unless stated otherwise, we follow [18] for terminology and definitions.

By a (d,c,r) graph, we mean an r-regular graph of diameter d and connectivity c.

By a *dangler* at w, we mean that part of an edge hanging from w, waiting to be connected to some other vertex to form an edge. In other words, an edge (w,z) can be created by fusing a dangler at w and a dangler at z, or, an edge (w,z) can be split into two danglers, one at w and the other at z.

$$\text{Let ODD}(x) = \begin{cases} 1 & \text{for } x \text{ odd,} \\ 0 & \text{for } x \text{ even,} \end{cases}$$

$$N(u) = \{v | (u,v) \in E(G)\}$$

and

$$\mu(d,c,r) = \min\{|V(G)| \,\|\, G \text{ is a } (d,c,r) \text{ graph}\}.$$

The functions $\mu(3,c,r)$ and $\mu(2,c,r)$ will be determined in Section 2, and $\mu(d \geqslant 4, c,r)$ will be determined in Section 3. In Section 4, the results are consolidated and modified to get upper bounds for c and d.

We assume $r > c \geqslant 1$ throughout our discussion. Even though we have excluded the case $r = c$, it can be seen that our method can be easily adapted to this case with slight modifications.

2. GRAPHS WITH DIAMETER AT MOST THREE

First, we determine $\mu(3,c,r)$ and then consider the case $d = 2$.

Theorem 1. $\mu(3,c,r) = 2r + 2$.

Proof. Let $d = 3$.

Suppose $c = 1$. Let w be a cutvertex on an endblock of G. Since $d = 3$, there exists at least one endblock of G (containing w) such that w is adjacent to all the other vertices of this block. This endblock has less than r vertices and hence degree of any noncutvertex of this block is $\leqslant r$, a contradiction. Hence let $c \geqslant 2$.

Take $V(G) = \{u,v,N_1,N_2\}$, where $N_1 = \{1,2, \ldots ,r\}$ and $N_2 = \{1',2', \ldots ,r'\}$. Make $\{u\} \cup N_1$ and $\{v\} \cup N_2$ as complete graphs. Introduce the edges $\{(i,i')|1 \leqslant i \leqslant c\}$. If c is even, delete the edges $(1,2),(3,4), \ldots ,(c-1,c)$, $(1',2'),(3',4'), \ldots ,((c-1)',c')$, and, if c is odd, introduce the edge $(1,2')$ and delete the edges $(1,2),(1,3),(4,5),(6,7), \ldots ,(c-1,c),(1',2'),(2',3')$, $(4',5'), \ldots ,((c-1)',c')$. This gives a $(3,c,r)$ graph and the minimality of the number of vertices is obvious since $|N(u)| = |N(v)| = r$. ■

Let $d = 2$ for the rest of this section.

In this case, first it is shown that the connectivity c is $\geqslant 3$. Next, it is proved that when $c = 3$, the regularity r cannot be > 6, and the minimum order of (2,3,4),(2,3,5), and (2,3,6) graphs are determined. In the case where $c \geqslant 4$, a necessary condition in the form of an inequality having a quadratic expression is obtained. Three different subcases arise depending on the roots of the corresponding quadratic equation and the required graphs will be constructed in each of these cases.

The following notation is used in this section. Let $V(G) = C \cup X \cup Y$ where C is a cutset of G with c vertices, X is the union of some of the components of $G - C$, Y is the union of the remaining components of $G - C$, and both X and Y are nonempty. Let $|X| \leqslant |Y|$ for the sake of definiteness. Note that there cannot be any edge between a vertex of X and a vertex of Y. Since $d = 2$, every vertex in X or Y is adjacent to at least one vertex of C. Hence $|X| \geqslant r - c + 1$. Let CX (CY) denote the number of edges between C and X (Y). Hence $CX + CY \leqslant cr$; $CX \geqslant |X|$; $CY \geqslant |Y|$.

Lemma 1. There does not exist a $(2,c,r)$ graph for $c \leqslant 2$ and $r > c$.

Proof. It is easy to verify that a $(2,1,r)$ graph does not exist. Suppose G is a $(2,2,r)$ graph. Since (the vertices of) C can be adjacent to at most $cr = 2r$ vertices of $X \cup Y$, we have $|X \cup Y| \leqslant 2r$. Since $|Y| \geqslant |X| \geqslant r - c + 1 = r - 1$, we have $|X| = r - 1$ or r.

If $|X| = r - 1$, then X is complete and every vertex of X is adjacent to both the vertices of C and hence $r - 1 \leqslant |Y| \leqslant CY \leqslant 2r - 2(r - 1) = 2$. This implies $r = 3$ (since we consider only the case $r > c$) and $|Y| = 2$. Hence Y is complete and every vertex of Y is adjacent to both the vertices of C. This gives $CY = 4$, a contradiction.

If $|X| = r$ then $|Y| = r$ and since $CX + CY \leqslant 2r$, any vertex in $X \cup Y$ is adjacent to exactly one vertex of C. This implies that there exist $x \in X$ and $y \in Y$ such that $d(x,y) > 2$, a contradiction.

Hence G cannot be a $(2,2,r)$ graph. ∎

Lemma 2. If G is a $(2,3,r)$ graph then $r \leqslant 6$.

Proof. It can be easily verified that when $r = 4,5$, or 6, μ is 7,12, and 13 respectively. The corresponding graphs can be constructed having $2 + 2$, $3 + 6$, and $4 + 6$ vertices in $X \cup Y$. See Figure 1 for examples. In the figures, a rectangle represents a complete graph on the vertices inside it and a dotted line represents an edge which is deleted to get the required graph.

Let $r \geqslant 7$. We have $|X| \geqslant r - c + 1 = r - 2$.

Case 1. Let $|X| = r - 2$.

This implies that every vertex of X is adjacent to every other vertex of X and hence

$$r - 2 = r - c + 1 \leqslant |Y| \leqslant CY \leqslant c(r - r + c - 1) = 6.$$

This gives $r \leqslant 8$.

Suppose $r = 7$. This means $|Y| \geqslant 5$. If $|Y| = 5$ then every vertex of Y is adjacent to every vertex of C and hence $CY = 15$, a contradiction to $CY \leqslant 6$. Similarly if $|Y| = 6$, we have $CY \geqslant 12$, a contradiction.

Suppose $r = 8$. This implies $|Y| = CY = 6$. But $|Y| = 6$ implies $CY \geqslant 18$ $(= |Y|(r - |Y| + 1))$, a contradiction, and case 1 is impossible.

Case 2. Let $|X| = r - 1$.

Here $CX \geqslant 2(r - 1)$ and $CY \leqslant 3r - 2(r - 1) = r + 2$. Hence $r - 1 \leqslant |Y| \leqslant r + 2$.

If $|Y| = r - 1$ then $CY \geqslant 2(r - 1)$ and this implies $2(r - 1) \leqslant r + 2$. Hence $r \leqslant 4$, a contradiction.

If $r \leqslant |Y| \leqslant r + 2$, then at least $r - 2$ (that is, $\geqslant 6$) vertices of Y are adjacent to exactly one vertex of C (since $CY \leqslant r + 2$). If a vertex $y \in Y$ is adjacent to exactly one vertex of C, say, c_1, then c_1 should be adjacent to all the $r - 2$ vertices of X, because $d = 2$, and hence c_1 is adjacent to no other vertex of Y. Since $c = 3$, C can accommodate at most 3 such vertices from Y and this implies $|Y| \leqslant 3$, a contradiction. Hence case 2 is impossible.

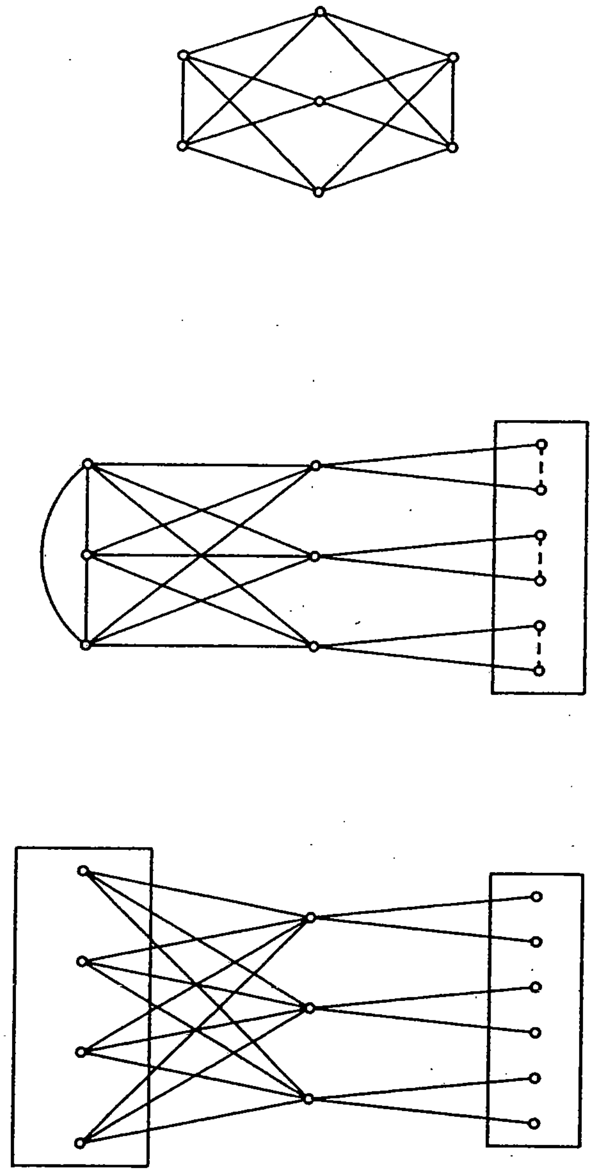


FIG. 1.

Case 3. Let $|X| \geqslant r$.

Here $|Y| \geqslant r$. As we have just seen, if a vertex of X (Y) is adjacent to exactly one vertex of C, say c_1, then c_1 must be adjacent to all the vertices of Y (X) and this gives deg $c_1 > r$, a contradiction. Hence every vertex in $X \cup Y$ is adjacent to at least two vertices of C and hence $CX + CY \geqslant 4r$, a contradiction to $CX + CY \leqslant 3r$. This implies that case 3 is also not possible.

This completes the proof of the Lemma. ∎

In view of Lemmas 1 and 2, let $c \geqslant 4$.

Let G be a $(2,c,r)$ graph. Let $|X| = r - c + 1 + k$ and $|Y| = r - c + 1 + m$, where $k,m \geqslant 0$. Our aim is to minimize $k + m$. We have

$$CX \geqslant (r - c + 1 + k)(r - (r - c + k));$$

$$CY \geqslant (r - c + 1 + m)(r - (r - c + m)).$$

Let $x = r - 2c + 2$ *and* $y = k + m$.

Now $cr \geqslant CX + CY$ gives on simplification

$$y^2 + y(x - 1) - cx - 2km \geqslant 0 \tag{1}$$

Note that it is necessary for k and m to satisfy (1), but just this alone is not sufficient to construct a graph with these values of k and m.

It is obvious that $y = k = m = 0$ satisfy (1) when $x \leqslant 0$, that is when $r \leqslant 2c - 2$. We shall construct a $(2,c,r)$ graph on $2r - c + 2$ vertices (that is $y = 0$) when $c \geqslant 4$, $r \leqslant 2c - 2$, and cr is even, and on $2r - c + 3$ vertices when $c \geqslant 4$, $r \leqslant 2c - 2$, and cr is odd. An extra vertex has to be added when c and r are odd, since there cannot exist a regular graph of odd degree on an odd number $(2r - c + 2)$ of vertices.

Let cr be even.

Let $|X| = |Y| = r - c + 1$ and $|C| = c$. Make X and Y complete separately. Add all possible edges between X and C, and between Y and C. On the vertices of C construct a regular graph of degree $2c - 2 - r$. Since at least one of c and $2c - 2 - r$ is even such a graph can be easily constructed as in Harary [8] and the resulting graph is the required $(2,c,r)$ graph.

Since the construction of an s-regular graph on t vertices will be used later with some modifications, we describe it here. The additions in the following are to be taken as modulo t.

Let the t vertices be $\{0,1,2, \ldots ,t - 1\}$. If s is even, join i to $i + j$ where $1 \leqslant j \leqslant s/2$ and $0 \leqslant i \leqslant t - 1$. If s is odd, (then t will be even), join i to $i + j$, where $1 \leqslant j \leqslant \lfloor s/2 \rfloor$ or $j = t/2$ and $0 \leqslant i \leqslant t - 1$. In these graphs the edges $(0,1),(2,3),(4,5), \ldots$, give a set of independent edges and $(0,\lfloor s/2 \rfloor + 1)$, $(1,\lfloor s/2 \rfloor + 2), \ldots$, give a set of independent edges which are not present in the constructed graph.

Let cr be odd. This gives $r \leqslant 2c - 3$ instead of $r \leqslant 2c - 2$.

Let $|X| = r - c + 1$ and $|Y| = r - c + 2$. Make X and Y complete. Add all possible edges between X and C. Let $V(Y) = \{y_i\}$ and $V(C) = \{c_j\}$. Join y_i to every vertex of C except c_i. Since $r \leqslant 2c - 3$, we have $c \geqslant r - c + 3$ and hence at this stage of construction, say H,

$$\deg_H c_i = \begin{cases} 2r - 2c + 2 & \text{for } i \leqslant r - c + 2 \\ 2r - 2c + 3 & \text{for } i > r - c + 2 \end{cases}$$

Consider a regular graph of degree $2c - 3 - r$ on c vertices. Introduce $(r - c + 2)/2$ independent new edges. This gives a graph on c vertices, with $r - c + 2$ vertices of degree $2c - 2 - r$ and the remaining vertices of degree $2c - 3 - r$. Imposing this graph on H such that every vertex of C is of degree r, we get the required $(2,c,r)$ graph. We have proved

Lemma 3. $\mu(2, c \geq 4, r \leq 2c - 2) = 2r + 2 - c + \mathrm{ODD}(cr)$.

Let $c \geq 4$ and $r > 2c - 2$ (that is $x \geq 1$) henceforth.

Let $f(y) = y^2 + y(x - 1) - cx$.

$$(1) \text{ implies } f(y) \geq 0 \tag{2}$$

Let s be the least nonnegative integer satisfying $f(s) \geq 0$. Hence $s = \lceil(-(x - 1) + \sqrt{(x - 1)^2 + 4cx})/2\rceil$ (since $x \geq 1$ implies that one root of $f(y) = 0$ is negative and the other positive).

Note that $s = 0$ if we consider $c \leq r \leq 2c - 2$, that is when $x \leq 0$, and $s = c$ when $r > c(c - 1)$, because in this case $f(c - 1) < 0$ and $f(c) > 0$.

Lemma 4. Suppose there exists a $(2,c,r)$ graph G with $k = 0$ and $m = s$. Then G has the minimum number of vertices; that is $\mu(2,c,r) = |V(G)|$.

Proof. Suppose $\mu(2,c,r) \leq |V(G)| = 2r + 2 - c + s$. Then there exist some $k_1,m_1 \geq 0$ such that $t = k_1 + m_1 < s$ and $t^2 + t(x - 1) - cx \geq 2k_1m_1 \geq 0$. By the definition of s, we have $s \leq t$, a contradiction. ∎

Lemma 5. Let $r(c + s)$ be odd. Suppose there exists a $(2,c,r)$ graph G with $k = 0$ and $m = s + 1$. Then $|V(G)| = \mu(2,c,r)$.

Proof. The extra vertex is added to make $|V(G)|$ $(= 2r - c + s + 3)$ an even number, since r is odd. The proof of this lemma is similar to that of the previous one. ∎

Lemma 6. A graph G as described in either of the previous two lemmas exists iff $r \leq c(c - 1)$.

Proof. Suppose there exists a $(2,c,r)$ graph with $k = 0$ and $m = s$ or $s + 1$. Since $k = 0$, we have $CX = (r - c + 1)c$ and hence $r - c + 1 + m = |Y| \leq CY \leq c(c - 1)$.

That is,

$$m \leq c^2 - r - 1. \tag{3}$$

We know

$$(-(x - 1) + \sqrt{(x - 1)^2 + 4cx})/2 \leq s \leq m. \tag{4}$$

Combining (3) and (4) and simplifying, we get $r \leq c(c - 1)$.

To prove the sufficiency, let $2c - 2 < r \leq c(c - 1)$. The required graph is constructed as follows.

Let $|C| = c$, $|X| = r - c + 1$ and

$$|Y| = \begin{cases} r - c + 2 + s & \text{for } r(c + s) \text{ odd} \\ r - c + 1 + s & \text{for } r(c + s) \text{ even.} \end{cases}$$

Starting from $r \leq c(c-1)$ and working backward in the simplification carried out proceeding from (3) and (4), we get

$$(-(x-1) + \sqrt{(x-1)^2 + 4cx})/2 \leq c^2 - r - 1.$$

Since the RHS is an integer, the above implies $s \leq c^2 - r - 1$. Hence

$$r - c + 1 + s \leq c(c-1). \tag{5}$$

If $r(c+s)$ is odd then the LHS of (5) is odd, and since the RHS of (5) is even, we have $r - c + 2 + s \leq c(c-1)$ in this case. This along with (5) implies $|Y| \leq c(c-1)$.

It is easy to check that $f(-cx) > 0, f(0) < 0, f(c) > 0$, and $f(z) > 0$ for z not in the interval given by the roots of $f(y) = 0$. When $r \leq c(c-1)$, we have $f(c-1) \geq 0$ and hence $s \leq c-1$. This gives $|Y| \leq r+1$, equality holding only when $s = c-1$ and r is odd.

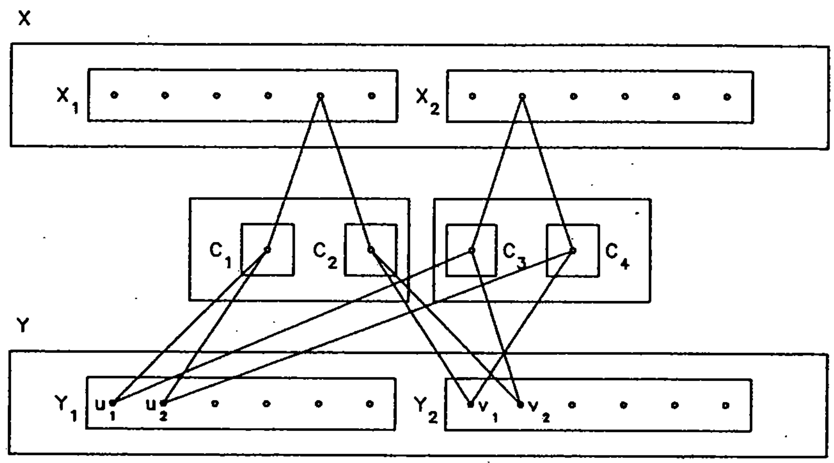
Suppose $|Y| = r+1$ is an even number. Complete $X \cup C$ and Y. Delete a perfect matching from Y (that is, from edges with both the endpoints in Y). Delete $(r+1)/2$ edges from C, deleting at least one edge from each vertex (such a deletion is possible since $c(c-1)/2$ edges are available within C; $c(c-1)/2 \geq r/2$; since r is odd and LHS is an integer we have $c(c-1)/2 \geq (r+1)/2$; and since $r \geq 2c-2$, $(r+1)/2 \geq c$). Join each vertex of Y to exactly one vertex of C, so that the degree of every vertex of C becomes r. The resulting graph is the required graph.

Suppose $|Y| \leq r$. Complete X and Y. Introduce all the edges between X and C. This implies that it is necessary to have $CY \leq c(c-1)$, and this is assured as follows. Since we have taken $k = 0$ and $m = s$ or $s+1$, we have $y = k + m \geq s$ and hence $f(y) \geq 0$. This is the same as $cr \geq CX + CY$ and this implies $CY \leq c(c-1)$.

Join every vertex of Y to $p = r + 1 - |Y|$ $(= c - s - \text{ODD } r(c+s) \leq c)$ vertices of C in a cyclic fashion. That is, taking $V(C) = \{0,1,2,\ldots,c-1\}$, $V(Y) = \{y_i\}$, join y_1 to $\{0,1,2,\ldots,p-1\}$, y_2 to $\{p,p+1,\ldots,2p-1\}$, y_3 to $\{2p,2p+1,\ldots,3p-1\}$, etc., where the vertices of C are taken modulo c. In this way, q degrees for the vertices $\{0,1,2,\ldots,a\}$ for some $a \leq c-1$, and $q-1$ degrees for the remaining vertices $\{a+1,a+2,\ldots,c-1\}$ of C are used for some $q \leq r$ ($q \leq r$ is assured by $CY \leq c(c-1)$). Hence, now we have to increase the degrees of the vertices $\{0,1,2,\ldots,a\}$ by $t (= r - q \geq 0)$ and the degrees of the vertices $\{a+1,a+2,\ldots,c-1\}$ by $t+1$, to get an r regular graph. This is done by imposing graph H on C as follows.

The graph H should have $a+1$ vertices of degree t and $c-a-1$ vertices of degree $t+1$. Consider $\Sigma_{v\in H} \deg_H v = (a+1)t + (c-a-1)(t+1) = c(c-1) - |Y|(r+1-|Y|)$. If $|Y|$ is odd then necessarily r is even by the definition of $|Y|$, and hence $\Sigma_{v\in H} \deg_H v$ is even. This assures that in the degree requirements for H, the number of odd degree vertices is even. Consider a regular graph of degree m on c vertices, where m is even and m is t or $t+1$. If $m = t$, introduce $(c-a-1)/2$ new independent edges; if $m = t+1$, delete $(a+1)/2$ independent edges already present. The resulting graph is H. H is imposed on the vertices of C in the obvious way to get the required r regular graph. ∎

Combining Lemmas 4, 5, and 6, we have,



(a) c=4; n=13

FIG. 2A.

Lemma 7. $\mu(2, c \geqslant 4, 2c - 2 < r \leqslant c(c-1)) = 2r + 2 - c + s + \text{ODD}\, r(c+s)$.

Lemma 8. $\mu(2, c \geqslant 4, r > (c(c-1)) = 2r + 2$.

Proof. When $r > c(c-1)$, we have $f(c-1) < 0$ and $f(c) > 0$. Hence $s = k + m = c$ and this implies $\mu \geqslant 2r + 2$. Graphs achieving this bound will be constructed which will prove the lemma. See Figure 2 for examples of this construction.

Case 1. Let $r - \lfloor c/2 \rfloor + 1$ be even.

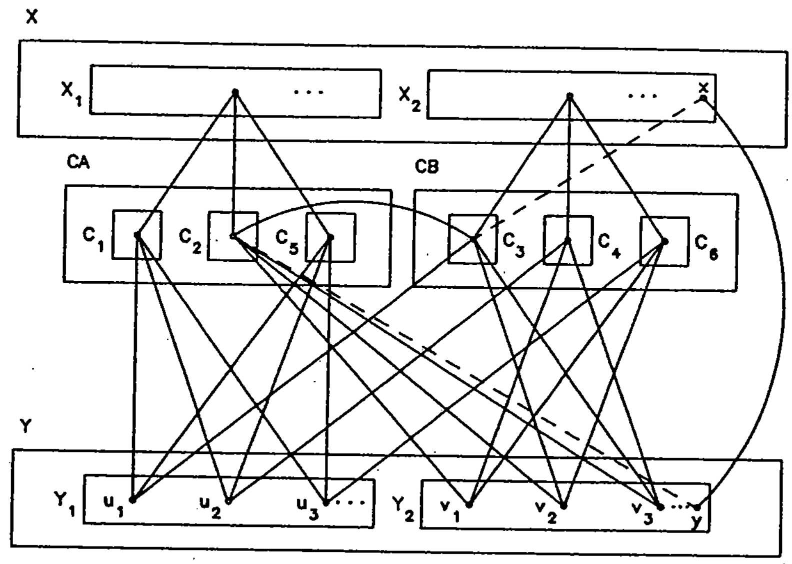
Let $|X| = |Y| = r - \lfloor c/2 \rfloor + 1$. Let $X = X_i \cup X_2$, where each X_i has exactly half the vertices of X. Let $Y = Y_1 \cup Y_2$ be defined similarly. Let $C = \bigcup_{i=1}^{6} C_i$, where C_i's are mutually disjoint and

$$|C_i| = \begin{cases} \lfloor c/4 \rfloor & \text{for } 1 \leqslant i \leqslant 4 \\ 0 & \text{for } 5 \leqslant i \leqslant 6 \text{ and } \lfloor c/2 \rfloor \text{ even} \\ 1 & \text{for } 5 \leqslant i \leqslant 6 \text{ and } \lfloor c/2 \rfloor \text{ odd.} \end{cases}$$

Hence

$$|C| = \begin{cases} c & \text{for } c \text{ even} \\ c - 1 & \text{for } c \text{ odd.} \end{cases}$$

Complete $X, Y, CA = C_1 \cup C_2 \cup C_5$ and $CB = C_3 \cup C_4 \cup C_6$. Introduce all possible edges between X_1 and CA, and between X_2 and CB. The edges between C and Y are to be introduced in such a way that the distance between the vertices of X and Y or between the vertices of C is $\leqslant 2$. Since $r - \lfloor c/2 \rfloor + 1 = |Y| > 9$, we have $|Y_i| > 4$, for $i = 1,2$. Let $u_1, u_2, u_3 \in Y_1$ and $v_1, v_2, v_3 \in Y_2$. Join



(b) c=7; n=44

FIG. 2B.

$$u_1 \quad \text{with } C_1 \cup C_3 \cup C_5,$$

$$v_1 \quad \text{with } C_2 \cup C_4 \cup C_6,$$

$$u_2 \quad \text{with } C_1 \cup C_4 \cup C_5,$$

and

$$v_2 \quad \text{with } C_2 \cup C_3 \cup C_6.$$

In case C_5 and C_6 are nonempty, for some $w \in C_2$, join

$$u_3 \quad \text{with } C_1 \cup C_5 \cup C_6 \cup (C_2 - w)$$

and

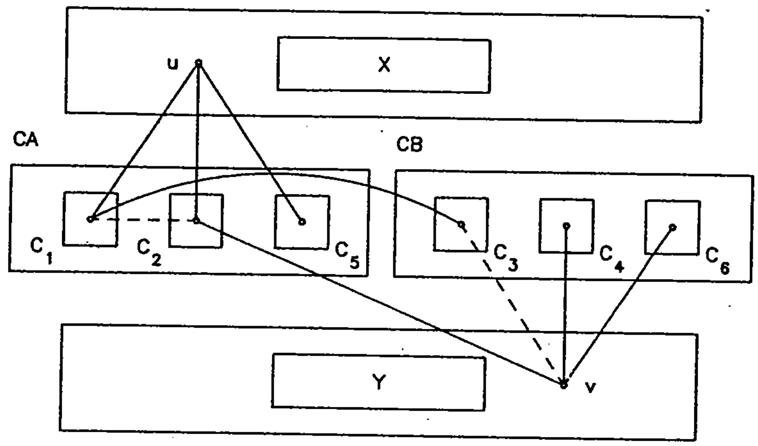
$$v_3 \quad \text{with } \{w\} \cup C_3 \cup C_4.$$

These connections ensure the distance requirements for the vertices within C. Now, the edges at the remaining vertices of Y_1 are defined similar to those of u_1 and for the remaining vertices of Y_2 they are as those of v_1. This ensures the distance requirements between the vertices of X and Y.

The resulting graph is a $(2, 2\lfloor c/2 \rfloor, r)$ graph on $2r + 2$ vertices. If C is even, this is the required graph. If C is odd, this graph is modified as follows.

Let $x \in X_2$, $y \in Y_2 - \{v_1,v_2,v_3\}$, $w_2 \in C_2$, and $w_3 \in C_3$. Introduce the edges (x,y) and (w_2,w_3), and delete the edges (x,w_3) and (y,w_2). Now, instead of C, the set

$C \cup \{x\}$ or $C' \cup \{y\}$ is a minimum cut with c number of vertices and this modified graph is the required graph



c=7; n=45

FIG. 3.

In Figure 2, the edges at many of the vertices are not shown, and for these vertices the pattern of connection will be similar to that of a vertex whose connections are shown.

Case 2. Let $r - \lfloor c/2 \rfloor + 1$ be odd.

Construct a $(2,c,r-1)$ graph on $2r$ vertices as in case 1. Note that the assumption $r > c(c-1)$ was used in that construction only to get $|Y_i| > 4$. Since this inequality holds good for $r-1$ also, such a construction is possible and gives a $(2,c,r-1)$ graph.

Introduce two new vertices u and v. Join u with every vertex in $X \cup CA$ and join v with every vertex in $Y \cup CB$. Now the degree requirements are satisfied for all the vertices, but not the distance requirements. Let $w_1 \in C_1$, $w_4 \in C_2$, and $w_5 \in C_3$. Introduce the edges (v,w_4) and (w_1,w_5). Delete the edges (w_1,w_4) and (v,w_5). It is easy to verify that the modified graph is a $(2,c,r)$ graph. ∎

Figure 3 shows only the modification to be carried out, starting from the graph of Figure 2b.

It is interesting to note that the formulas obtained in all the cases considered here for $d = 2$ and the one $\mu(2,c \geq 2,c) = c + 2 + \text{ODD } c$ given in [1] can be put in a single formula. Let s be defined as follows for all values of c and r. Let s be the least nonnegative integer satisfying $f(y) \geq 0$, where $f(y) = y^2 + y(x-1) - cx$ and $x = r - 2c + 2$. Noting that $s = 0$ if $c \leq r \leq 2c - 2$ and $s = c$ if $r > c(c-1)$, we have

Theorem 2. Suppose $r = 2$ when $c = 2$, and $r \leq 6$ when $c = 3$. Then, $\mu(2,c \geq 2, r \geq c) = 2r + 2 - c + s + \text{ODD } r(c+s)$.

3. GRAPHS WITH DIAMETER AT LEAST FOUR

Let $d \geqslant 4$, $c \geqslant 1$, and $r > c$.

In this section, we distinguish two special vertices u and v in the graph G, such that $d(u,v) = d$. Then, let

$$N_i = \{w|d(w,u) = i\} \qquad \text{for } 0 \leqslant i \leqslant d-2$$

$$N_{d-1} = \{w|d(w,u) \geqslant d-1 \text{ and } w \neq v\}$$

and

$$N_d = \{v\}.$$

The definitions of N_{d-1} and N_d are altered from the usual definition of the neighborhoods in order to make our discussions easier.

Since any vertex in N_i can be adjacent to vertices of N_{i-1}, N_i, and N_{i+1} only, we get

$$|N_{i-1}| + |N_i| + |N_{i+1}| \geqslant r+1 \qquad \text{for } 1 \leqslant i \leqslant d-1 \tag{1}$$

and

$$|N_1| \geqslant r,\ |N_{d-1}| \geqslant r. \tag{2}$$

Also

$$|N_i| \geqslant c \qquad \text{for } 2 \leqslant i \leqslant d-2. \tag{3}$$

The number of vertices in each N_i will be first fixed to be the minimum, satisfying the above inequalities. The conditions under which this number of vertices will give the required graph will be studied. The problem arises when the connections between the consecutive N_i's could not be fixed satisfying all the requirements and in these cases the number of vertices is not enough. Some more vertices will be added in these cases, making sure that the number of such vertices added is minimum. Usually, the problem is in the connections between N_1 and N_2 in one end, and possibly between N_{d-2} and N_{d-1} in the other end. The connections between the other consecutive N_i's are fixed easily. We show that, when there is a problem at both ends, an adjustment in one end would result in a chain reaction of edge adjustments and possibly introduction of some new vertices. This chain reaction stops only when it reaches the other end and solves the problem of that end also. When the problem arises in exactly one end, it is solved by the introduction of just one vertex, and no chain reaction occurs.

The constructions naturally split into two major cases. When $r < 3c$, we can start with N_i's, $2 \leqslant i \leqslant d-2$, each having exactly c vertices, but this will not be the case when $r \geqslant 3c$.

Case 1. Let $r < 3c$.

If $c = 1$ then $r = 2$. Since a $(d,1,2)$ graph does not exist let $c \geqslant 2$.

Let $|N_1| = |N_{d-1}| = r$ and $|N_i| = c$ for $2 \leqslant i \leqslant d-2$. The inequalities (1), (2), and (3) are satisfied and the total number of vertices at this stage is $2r + 2 + (d-3)c$.

Complete each N_i and add all the edges between u and N_1, and between N_{d-1} and v. Now the interconnections between the consecutive N_i's are to be given.

By the statement "let (s,t) be the connection at N_i," we mean that every vertex of N_i is adjacent to s vertices of N_{i-1} and t vertices of N_{i+1}. The connection between N_i and N_{i+1} is done in the circular way, as given below. Let the vertices of N_i $(2 \leqslant i \leqslant d-2)$ be labeled as $0,1,2,\ldots$. We are using the same labels for vertices of different sets, but this is being done deliberately to avoid cumbersome notation. We specifically mention to which set a label belongs, whenever it is not clear from the context. The vertex j of N_i is adjacent to $j, j+1, \ldots, j+t-1$ of N_{i+1}, where the numbers are taken modulo $|N_{i+1}|$. By a "connection scheme" we mean the sequence of connections at $N_2, N_3, \ldots, N_{d-2}$. Note that if (s,t) and (p,q) are two consecutive connections at N_i and N_{i+1} in a connecting scheme then $t = p$, provided $|N_i| = |N_{i+1}|$.

Let $s + t = r - c + 1$, where $|s - t| \leqslant 1$. Since $c < r < 3c$, we have $s + t \geqslant 2$ and $1 \leqslant s,t \leqslant c$. Consider the connection scheme $(s,t),(t,s),(s,t),\ldots$ ending with (s,t) if d is even and with (t,s) if d is odd. This gives all the interconnections between N_i and N_{i+1}, $2 \leqslant i \leqslant d-3$. Now only the connections between N_1 and N_2, and between N_{d-2} and N_{d-1} are to be given.

Since $N_1 \cup \{u\}$ is made complete in the beginning, degree of every vertex in N_1 is already r. So, to accommodate the edges from N_2, some edges of N_1 (that is, edges having both the endpoints in N_1) have to be deleted. Since removal of k edges from N_1 produces $2k$ danglers and accommodates $2k$ danglers from N_2, this implies that the number of edges between N_1 and N_2 (and similarly between N_{d-2} and N_{d-1}) must be even.

For the connecting scheme $(s,t),(t,s),(s,t),\ldots$, there are cs danglers at N_2 to be accommodated at N_1. Two other conditions to be satisfied by any connection between N_1 and N_2 are the following. Multiple edges should not be produced and at least c number of vertices of N_1 should be connected to N_2 by edges.

The following step accommodates all or most of the danglers from N_2 satisfying the above conditions.

Let $\{0,1,2,\ldots,r-1\}$ be the vertices of N_1. Let $\lfloor s/2 \rfloor = p$ and $\lfloor t/2 \rfloor = q$. Since $s < r - c + 1$, we have $r - p \geqslant c + p$.

Step A. If $p = 0$, nothing is performed in Step A. Let $p \geqslant 1$. The edges of N_1, joining $\{0,1,2,\ldots,p-1\}$ and $\{p,p+1,\ldots,r-1\}$ can be represented by a $p \times (r-p)$ array. Starting from the square $(0,p+i)$ and moving in the direction of the main diagonal until the square $(p-1, p+i+p-1)$ is reached, we get p squares which give an independent set of p edges, for each i, $0 \leqslant i \leqslant c-1$. (This is possible since $2p + c - 2 < r - 1$ and the square $(p-1, 2p+c-2)$ exists in the array.) By removing these p edges, accommodate $2p$ danglers from the vertex i of N_2, for $0 \leqslant i \leqslant c-1$. End of Step A.

Note that Step A can be carried out between N_{d-2} and N_{d-1} similarly.

We consider three major subcases at this stage. They are:

Without the introduction of new vertices,

1.1) the number of danglers from N_2 to N_1, and the number of danglers from N_{d-2} to N_{d-1}, both can be made even,

1.2) exactly one of the above two numbers can be made even, and

1.3) none of the two numbers can be made even.

Inclusion in any of these three categories depends on the values of d, c and r. The following table shows the relation between the values of d,c,r and the above classification.

Case	c	r	d	Type	ODD cr(d-1)
a)	even	—	—	1.1	0
b)	odd	odd	odd	1.1	0
c)	odd	odd	even	1.2	1
d)	odd	even, $r < 3c - 1$	—	1.1	0
e)	odd	even, $r = 3c - 1$	$d = 4$	1.1	0
f)	odd	$r = 3c - 1$	$d = 5$	1.1	0
g)	odd	$r = 3c - 1$	$d \geq 6$	1.3	—

Subcase a. c is even.

This implies cs is even. If s is even, Step A exhausts all the danglers at N_2. So, let s be odd. At the end of Step A, the vertex i of N_2 is not adjacent to $2p + i$ $(\leq 2p + c - 1 \leq r - t \leq r - 1)$ of N_1, and one dangler remains at each vertex of N_2.

Step B. Introduce the edges

$$\{(i, 2p + i) | 0 \leq i \leq c - 1\} \subseteq N_2 \times N_1$$

and delete the edges $(2p, 2p + 1), (2p + 2, 2p + 3), \ldots, (2p + c - 2, 2p + c - 1)$ from N_1 and this gives the required connection. End of Step B.

Similar procedure at the other end (between N_{d-2} and N_{d-1}) gives the required graph.

Subcase b. d, c, and r are all odd.

Here $s + t = r - c + 1$ is odd and hence in $\{s,t\}$ one is even and the other is odd. Let s be even. Since d is odd, the connecting scheme is $(s,t),(t,s), \ldots,(t,s)$. Since s is even, Step A gives the required graph.

Subcase d. c is odd, r is even, and $r < 3c - 1$.

Here $s + t$ is even, $s = t$ and both are even or both are odd. If both are even, the solution is obvious. Let both be odd. Since $r < 3c - 1$, we have $s + t < 2c$; $s,t < c$ and $s,t \leq c - 2$ (since c is odd). Split $r - c + 1$ as $s - 1$ and $t + 1$, instead of s and t. Both these numbers are even and $<c$. Consider the new connection scheme as $(s - 1, t + 1)$, $(t + 1, s - 1)$, ..., instead of (s,t), (t,s), ... This scheme is feasible since $s - 1, t + 1 < c$ and Step A at both ends gives the required graph.

Subcase e. c is odd, $r = 3c - 1$ is even and $d = 4$.

Here $s = t = c$ and $N_2 = N_{d-2}$.

Step C. Perform Step A, and then as in Step B, accommodate the even number of single danglers from the vertices $\{0,1, \ldots, c - 2\}$ of N_2 at N_1. Similarly accommodate such danglers from N_{d-2} $(= N_2)$ at N_{d-1}. End of Step C.

Now two danglers from $c - 1$ of N_2 are left out.

Step D. Delete the edge $(2c - 1, 2c)$ from N_{d-1} and introduce the edges $\{(c - 1, 2c - 1), (c - 1, 2c)\} \subseteq N_2 \times N_3$. End of Step D.

Since $2p + c - 1 < 2c - 1$, the vertex $c - 1$ of N_2 was not adjacent to the vertices $2c - 1$ and $2c$ of N_3, while performing Step D. Since c is odd and >1, we have $c \geqslant 3$, $s = t = c \geqslant 3$, $p \geqslant 1$ and hence during Step A, N_2 (N_{d-2}) is connected to at least c vertices of N_1 (N_{d-1}) directly by edges. This ensures that after Step D, $d = 4$ and the connectivity is c.

Subcase f. c is odd, $r = 3c - 1$, and $d = 5$.

After Step C, one dangler each at N_2 and N_3 are left out. Delete the edge $(c - 1, c - 1) \in N_2 \times N_3$, to get one more dangler each at N_2 and N_3. Now Step D performed at both the ends gives the required graph.

In this case also, $c \geqslant 3$ and this ensures that the diameter and connectivity do not change, though we have deleted one edge between N_2 and N_3.

Subcase c. c and r are odd, and d is even.

Here $s + t = r - c + 1$ is odd and hence for any combination of s and t one is even and the other odd. Let s be even and t be odd.

Since c is odd and $\geqslant 2$, we have $c \geqslant 3$. The number of vertices considered till now is $2r + 2 + c(d - 3)$, an odd number. Since there cannot exist an odd regular graph on odd number of vertices, at least one more vertex is to be added to construct the graph.

At the end of Step A at both ends, the connection between N_1 and N_2 is completed, and one dangler each from the vertices of N_{d-2} will be remaining. If $t = 1$, then $q = \lfloor t/2 \rfloor = 0$ and no connection would have been made between N_{d-2} and N_{d-1}; in this case, delete a hamiltonian cycle from N_{d-2}. This increases the number of danglers at each vertex of N_{d-2} to 3 ($\leqslant c$). Now $t = 3$ and $q = 1$, at N_{d-2}. Using Step A, accommodate two danglers from each vertex of N_{d-2}, and N_{d-2} will now be connected to at least c vertices of N_{d-1} by edges, this being essential to preserve the connectivity and diameter of the graph.

Step E. Introduce a new vertex w and the edges

$$\{(w,i) | 0 \leqslant i \leqslant c - 1\} \subseteq \{w\} \times N_{d-2}$$

and

$$\{(w,j) | q \leqslant j \leqslant q + r - c - 1\} \subseteq \{w\} \times N_{d-1}.$$

Delete the edges $S = \{(q, q + 1), (q + 2, q + 3), \ldots, (q + r - c - 2, q + r - c - 1)\}$ from N_{d-1}. End of Step E.

Since every edge in S has both endpoints with labels $\geqslant q$, they have not been deleted in Step A. The resulting graph is the required one. Note that N_{d-2} is still a minimum cut and the diameter is unchanged.

Subcase g. c is odd, $r = 3c - 1$, and $d \geqslant 6$.

Since $d - 3 \geqslant 3$, there exists an i such that $3 \leqslant i \leqslant d - 3$ and for every such i, $|N_{i-1}| = |N_i| = |N_{i+1}| = s = t = c$. This implies that every vertex x of N_i $(3 \leqslant i \leqslant d - 3)$ is adjacent to all the vertices of $N_{i-1} \cup (N_i - x) \cup N_{i-1}$, and s and t could not be adjusted as in Subcase d. Suppose an edge at $x \in N_i$, $3 \leqslant i \leqslant d - 3$, is deleted. To compensate for the loss in the degree at x, it should be made adjacent to a vertex of $N_{i-1} \cup N_i \cup N_{i+1}$ to which it was not adjacent previously. Since no such vertex exists, none of the edges at any vertex of N_i, $3 \leqslant i \leqslant d - 3$, can be deleted without introducing any new vertex.

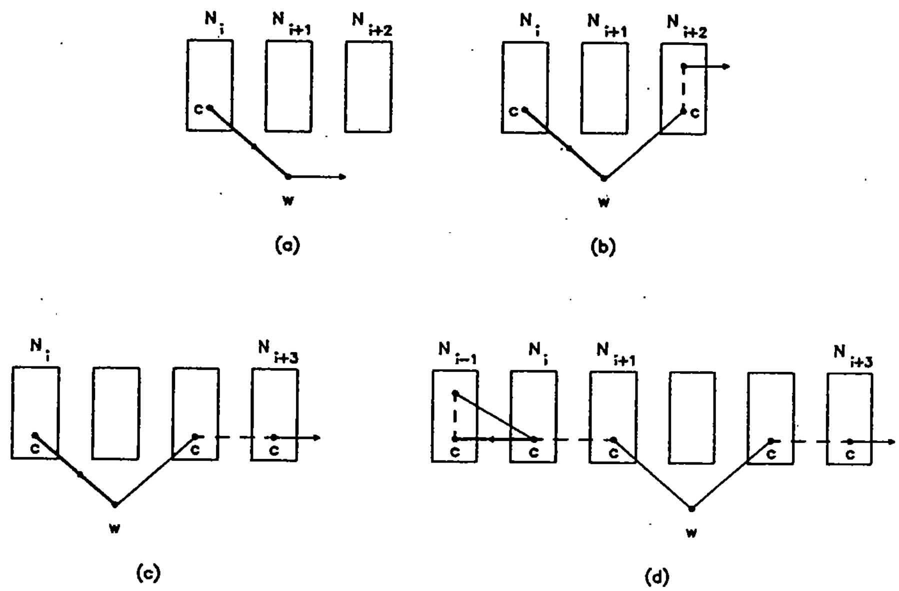


FIG. 4.

After carrying out Step C at both ends, we are left with a single dangler at $c - 1$ of N_2 and a single dangler at $c - 1$ of N_{d-2}. That the lonely dangler at N_2 (N_{d-2}) cannot be adjusted within $N_1 \cup N_2$ ($N_{d-2} \cup N_{d-1}$) is obvious. We have just noted that no modifications can be carried out using vertices of N_i, $3 \leqslant i \leqslant d - 3$, without introducing new vertices. Hence the graph cannot be constructed without introducing a new vertex.

Since the introduction of any number of new vertices always gives rise to an even number of danglers (since r is even), a lonely dangler can only be transferred from one place to the other and it cannot be destroyed unless it is paired with the other lonely dangler already present. To pair the lonely dangler at N_2 with the one at N_{d-2}, it has to be shifted to the right, by introducing new vertices. Note that any new vertex introduced should be adjacent to vertices in N_i, N_{i+1}, and N_{i+2} only, for some i, in order to maintain the diameter, and this new vertex acts as an element of N_{i+1} considering its adjacencies.

Suppose there is a lonely dangler at N_i and exactly one new vertex w is introduced. It can be shifted to w or to N_{i+2}, N_{i+3} or N_{i+4} to the right. (That it cannot be shifted any further to the right can be easily verified.) Figure 4 gives a schematic diagram depicting these cases. The new vertex w is adjacent to $\{0,1, \ldots ,c - 2\}$ of N_j, N_{j+1}, and N_{j+2}, where $j = i$ in the first three cases and $j = i + 1$ in the fourth case. The edges $\{(0,1),(1,2), \ldots ,(c - 3,c - 2)\}$ are deleted from N_j, N_{j+1}, and N_{j+2}. The adjustment with the remaining two danglers of w are shown in Figure 4. The thick arrow from N_i is the dangler being shifted and the thin arrow represents the new dangler being created. Dotted line represents an edge being deleted and an ordinary line represents a new edge being introduced.

Suppose $d = 6$, 7, or 8. By introducing one new vertex w, the lonely dangler at

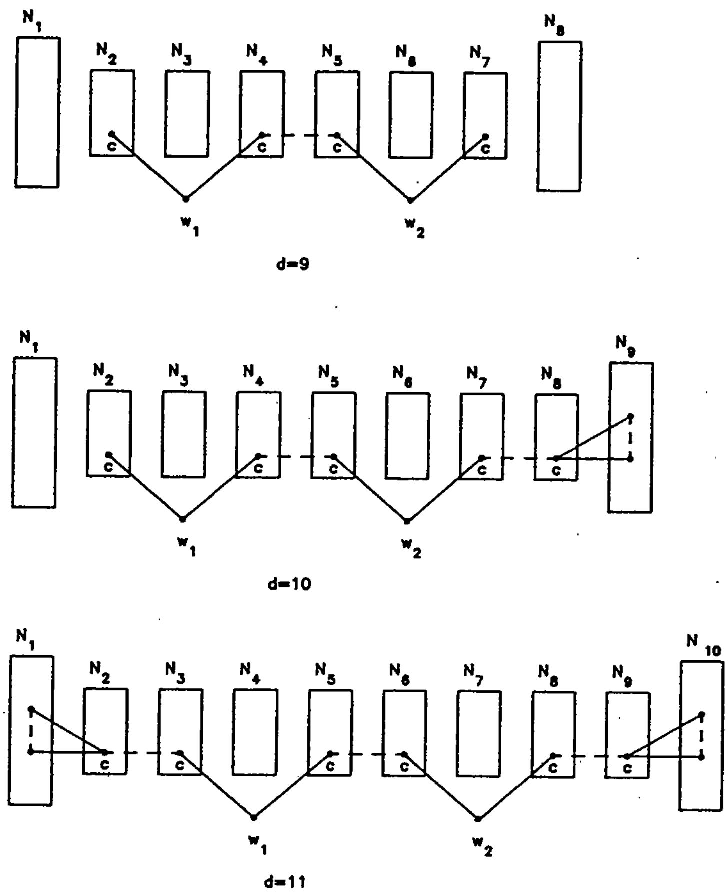


FIG. 5.

N_2 can be transferred to N_{d-2} by using case 2, 3, or 4. This dangler along with the one already available at N_{d-2} can be easily accommodated at N_{d-1}.

Let $d \geqslant 9$. By introducing a single new vertex, the lonely dangler at N_2 can be transferred at most to N_6, but this cannot be paired with the one at N_{d-2} without further addition of new vertices. Note that case 4 can be used only once, at $N_i = N_2$, since two new edges are being introduced between N_i and N_{i+1} in this case. Only the other cases can be used consecutively any number of times. This implies that a minimum of $\lceil (d - 5)/3 \rceil$ new vertices are to be introduced to construct the graph. Figure 5 gives the schematic diagram of such graphs for the cases $d = 9$, 10, and 11, and the other cases follow similarly. $s = c \geqslant 3$ implies that the connectivity and diameter do not change by any of the above steps in the construction.

This completes the construction in case 1.

Case 2. Let $r \geqslant 3c$.

Here $c \geqslant 1$.

Let $|N_1| = |N_{d-1}| = r$ and

$$|N_i| = \begin{cases} c & \text{for } 2 \leq i \leq d-2 \text{ and } i \bmod 3 = 0 \text{ or } 2 \\ r-2c+1 & \text{for } 2 \leq i \leq d-2 \text{ and } i \bmod 3 = 1. \end{cases}$$

Since equality holds in (2) and in (1) for $3 \leq i \leq d-3$, this number of vertices is essential to construct the graph.

Make each N_i complete. Add all possible edges between N_i and N_{i+1} for all i, $2 \leq i \leq d-3$. Join u (v) to every vertex of N_1 (N_{d-1}). As before, only the connections between N_1 and N_2, and between N_{d-2} and N_{d-1} are to be specified.

The number of danglers left at N_2 is $c(r - 2c + 1)$ and at N_{d-2} it is c^2 or $(r - 2c + 1)c$ depending on whether d mod 3 is 1 or not. Let $s(t)$ be the number of danglers at each vertex of N_2 (N_{d-2}).

Subcase h. Let c be even.

At each end perform Step A or Steps A and B depending on whether s or t is even or odd. It can be easily checked that $c(r - 2c + 1), c^2 \leq r(r \leq 1)$; $r - p \geq c + p$ and $r - q \geq c + q$, where $p = \lfloor (r - 2c + 1)/2 \rfloor = \lfloor s/2 \rfloor$ and $q = \lfloor t/2 \rfloor$. Hence the above steps could be carried out and this gives the required graph.

Subcase i. Let c and r be odd.

Here $s = r - 2c + 1$ is even and Step A is enough at N_2.

If d mod 3 = 0, then $|N_{d-2}| = r - 2c + 1$, $t = c$, and hence use Steps A and B at N_{d-2}.

If d mod 3 = 2, then $|N_{d-2}| = c$, $t = r - 2c + 1$, and hence Step A is enough at N_{d-2}.

Let d mod 3 = 1. Here $|N_{d-2}| = t = c$. The number of vertices considered till now is $2r + 2 + \lfloor (d - 3)/3 \rfloor (r + 1) + c$, an odd number. Since r is odd, at least one more vertex is needed to construct a graph. If $c > 1$, then Steps A and E give the required graph. When $c = 1$, if we use Steps A and E, it will increase the diameter. Hence we proceed as follows.

Let $c = 1$ and $d = 4$.

If $r = 3$, then $c = 1$ implies that G has a bridge, say, (x,y). Since $d = 4$, at least one of x and y is adjacent to all the other vertices in the component containing it in $G - (x,y)$. This contradicts the regularity of G and hence $r \neq 3$. Since r is odd, $r \geq 5$. Instead of $(r - 1, 1)$, consider the connection $(r - 3, 3)$ at N_2. Since $r \geq 5$, we have $r - 3 \geq 2$. It is easy to check that the application of Steps A and E now gives the required graph.

Let $c = 1$ and $d \geq 7$. Here $|N_{d-3}| = r - 1$, an even number. Introduce a vertex w, join it to all the vertices of N_{d-3}, and delete a perfect matching from N_{d-3}. Now one dangler from w and one dangler from the unique vertex of N_{d-2} are remaining and they are accommodated at N_{d-1}. This preserves the diameter. Since N_2 is still a cut vertex, the connectivity remains one.

Subcase j. Let c be odd and r be even.

Note that when $c = 1$, r has to be >2. Hence $r \geq 4$.

If $d = 4$, consider the connection $(r - 2c, c + 1)$ at N_2. Now Step A at both ends gives the required graph.

If $d = 5$ and $c > 1$, construct as in Subcase *f*. If $c = 1$, the method of Subcase *f* will not work, since there is exactly one edge between N_2 and N_3. Also $|N_2| = |N_3| = 1$

implies that there should be an odd number of edges connecting N_1 and N_2 or N_3 and N_4, which is impossible. Hence at least one more vertex is to be added to construct the graph. Take $|N_2| = 1$ and $|N_3| = 2$. Complete N_3 and consider the connections $(r - 2, 2)$ at N_2 and $(1, r - 2)$ at N_3. Now Step A at both the ends give the required graph.

Let $d \geqslant 6$.

Redefine the number of vertices in N_i, $2 \leqslant i \leqslant d - 2$ as

$$|N_i| = \begin{cases} c & \text{for } i \bmod 3 = 2 \\ c + 1 & \text{for } i \bmod 3 = 0 \\ r - 2c & \text{for } i \bmod 3 = 1. \end{cases}$$

Complete each N_i and add all possible edges between N_i and N_{i+1} for $2 \leqslant i \leqslant d - 3$. The number of danglers left at N_2 is $c(r - 2c)$ and at N_{d-2} it is $c(c + 1)$ or $(c + 1)(r - 2c)$ or $(r - 2c)c$, an even number in any case. These danglers are accommodated as usual. If $d \bmod 3$ is 0 or 1, this modification does not change the total number of vertices. If $d \bmod 3 = 2$, then exactly one more vertex is added. We show that this is essential.

So, let $d \bmod 3 = 2$. Suppose exactly $\lfloor (d - 3)/3 \rfloor (r + 1) + 2c$ vertices are allowed in $\cup_{r=2}^{r-2} N_i$. It is easy to see that the only sequence of $d - 3$ integers $\geqslant c$, with at least one of them equal to c, such that the sum of every three consecutive elements is $\geqslant r + 1$ and the total sum as above is

$$c,c,r - 2c + 1,c,c,r - 2c + 1, \ldots ,c,c,r - 2c + 1,c,c.$$

This is the allotment with which we started and in this case the number of danglers left at N_2 or N_{d-2} is $c(r - 2c + 1)$, an odd number. Steps A and B will leave two lonely danglers one at N_2 and the other at N_{d-2}. These two cannot be brought together without an increase in the number of vertices, as noted in Subcase g. Hence the introduction of at least one more vertex is essential.

Note that we are able to construct the graph in this case with just one extra vertex, whereas in Subcase g a number of new vertices are to be added depending on d. The reason is that, in this case, the number of vertices in some N_i's could be reduced whereas it was not possible in Subcase g. We have proved:

Theorem 3.

$$\mu(d \geqslant 4, c \geqslant 1, c < r < 3c) = \begin{cases} 2r + 2 + c(d - 3) + \lceil (d - 5)/3 \rceil, \\ \quad \text{for } d \geqslant 6,\ c \text{ odd},\ r = 3c - 1, \\ 2r + 2 + c(d - 3) + \mathrm{ODD}\ cr(d - 1), \\ \quad \text{otherwise.} \end{cases}$$

$$\mu(d \geqslant 4, c \geqslant 1, r \geqslant 3c) = \begin{cases} \lfloor (d + 3)/3 \rfloor (r + 1) + c(d \bmod 3) + 1, \\ \quad \text{for } (d \bmod 3 = 1,\ c = 1,\ r \text{ odd}), \text{ or} \\ \quad\quad (d = 5,\ c = 1,\ r \text{ even}), \text{ or} \\ \quad\quad (d \geqslant 8,\ d \bmod 3 = 2,\ c \text{ odd},\ r \text{ even}), \\ \lfloor (d + 3)/3 \rfloor (r + 1) + c(d \bmod 3), \\ \quad \text{otherwise.} \end{cases}$$

except for the combinations $(d,1,2)$ and $(4,1,3)$ which are not feasible.

4. SUMMARY

The following values for μ are available in the published literature [1,11,13–16]. We give them below for ready reference.

$$\mu(3k + j,1,3) = 2j + 4k, \qquad \text{where } 5 \leq j \leq 7$$

$$\mu(d,2,3) = 2d + 2$$

$$\mu(d,3,3) = \begin{cases} 3(d-1) + 2 & \text{for } d \text{ odd} \\ 3(d-1) + 3 & \text{for } d \text{ even.} \end{cases}$$

Let $4 \leq j \leq 6$ and $k \geq 0$.

$$\mu(3k + j,1,r) = \begin{cases} (k+2)(r+1) + 1 & \text{for } r \text{ even}, j = 4 \\ (k+2)(r+1) + 2 & \text{for } r \text{ odd}, j = 4 \\ (k+2)(r+1) + 3 & \text{for } r \text{ even}, j = 5 \\ (k+2)(r+1) + 2 & \text{for } r \text{ odd}, j = 5 \\ (k+3)(r+1) & \text{for } j = 6 \end{cases}$$

$$\mu(d,c,c) = \begin{cases} c(d-1) + 3 & \text{for } c \text{ odd}, d \text{ even} \\ c(d-1) + 2 & \text{otherwise.} \end{cases}$$

It can be verified that all the above formulas for $r > c$ agree with our formulas, though each one of the above is given in a different format. The formula for the case $c = r$ also fits in our formula obtained in the case $c < r < 3c$ and hence the same formula can be given for $c \leq r < 3c$.

Combining all the results we have:

Theorem 4. The combinations $(1, c, r \neq c)$, $(2, c \leq 3, r > c(c-1))$, $(4,1,3)$, and $(d,1,2)$ are infeasible and in the other cases μ is defined as follows, where $r \geq c \geq 1$.

$$\mu(1,c,c) = c + 1.$$

$$\mu(2,c,r) = 2r + 2 - c + s + \text{ODD } r(c + s).$$

$$\mu(3,c,r) = 2r + 2.$$

$$\mu(d \geq 4, c, r < 3c) = \begin{cases} 2r + 2 + c(d-3) + \lceil (d-5)/3 \rceil, \\ \quad \text{for } d \geq 6, c \text{ odd}, r = 3c - 1. \\ 2r + 2 + c(d-3) + \text{ODD } cr(d-1), \\ \quad \text{otherwise.} \end{cases}$$

$$\mu(d \geq 4, c, r \geq 3c) = \begin{cases} \lfloor (d+3)/3 \rfloor (r+1) + c(d \bmod 3) + 1, \\ \quad \text{for } (d \bmod 3 = 1, c = 1, r \text{ odd}), \text{ or} \\ \quad (d = 5, c = 1, r \text{ even}), \text{ or} \\ \quad (d \geq 8, d \bmod 3 = 2, c \text{ odd}, r \text{ even}). \\ \lfloor (d+3)/3 \rfloor (r+1) + c(d \bmod 3), \\ \quad \text{otherwise.} \end{cases}$$

It is expected that when d and r are kept constant and c is increased, μ will also increase monotonically, and this is the case when $d \geq 3$. But when $d = 2$, it is interesting to note that as c is increased in multiples of 1, μ decreases monotonically. (This can be easily seen since, when c is increased by 1 the increase in s is at most 1.)

Suppose we are interested in graphs having minimum number of vertices, with diameter 2, connectivity $\geq c$, and regularity r, then we will get graphs with $r + 2$ or $r + 3$ vertices which are r-connected. But this will not happen when $d \geq 3$. In this case the resulting graph will have connectivity c. This shows that only when $d = 2$, the approach by Klee and Quaife (connectivity $\geq c$) will give different results from those obtained by the others' approach (where connectivity $= c$).

Theorem 4 can be used to find an upper bound for the connectivity, given the number of vertices, diameter and regularity of a graph, or, to find an upper bound for the diameter, given the other three parameters.

Corollary 1. Let G be a (d,c,r) graph on n vertices. Then

$$c \leq r/3 \quad \text{for } n \leq 2r + 2 + (d - 3)r/3 \text{ and } d \bmod 3 = 0,$$

$$c \leq \min\{r/3, (n - \lfloor (d + 3)/3 \rfloor (r + 1))/(d \bmod 3)\}$$

$$\text{for } n \leq 2r + 2 + (d - 3)r/3 \text{ and } d \bmod 3 \neq 0;$$

$$c \leq \min\{r, (n - 2r - 2)/(d - 3)\} \text{ for } n > 2r + 2 + (d - 3)r/3.$$

Starting from these approximate bounds and using the values of μ given by Theorem 4 in the inequality $n \geq \mu$, better bounds for c can be obtained.

Corollary 2. For a (d,c,r) graph G on n vertices

i) $d \leq 3n/(r + 1)$, if $c \leq r/3$;

ii) $d \leq (n - 2r - 2)/c + 3$, if $c > r/3$;

iii) $d \leq (n - 2r - 2)3/(2r) + 3$, if G is a circulant.

Proof. (i) and (ii) follow directly from Theorem 11. (iii) follows from (ii) and from the fact that in a circulant $(2/3)r < c \leq r$ [5]. ∎

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Received November, 1986.
Accepted August, 1987.