Student Name: ______________________ Student ID # ______________________

UOSA Statement of Academic Integrity

On my honor I affirm that I have neither given nor received inappropriate aid in the completion of this exercise.

Signature: ______________________ Date: ______________________

Notes Regarding this Examination

Open Book(s) You may consult any printed textbooks in your immediate possession during the course of this examination.

Open Notes You may consult any printed notes in your immediate possession during the course of this examination.

No Electronic Devices Permitted You may not use any electronic devices during the course of this examination, including but not limited to calculators, computers, and cellular phones. All electronic devices in the student's possession must be turned off and placed out of sight (for example, in the student's own pocket or backpack) for the duration of the examination.

Violations Copying another's work, or possession of electronic computing or communication devices in the testing area, is cheating and grounds for penalties in accordance with school policies.

Question E1-1: Well-Formed Formulas (10 points)

Consider the following well-formed formula: $\neg(a \wedge b) \rightarrow (\neg a \vee \neg b)$

A. Is it satisfiable? *Justify* your answer.

B. Is it tautologous? *Justify* your answer.

C. Draw its circuit diagram.

Question E1-2: Well-Formed Formulas (10 points)

Consider the following well-formed formula: $((a \land (a \to b)) \lor c) \to (\neg b \land \neg c)$

A. Is it a contradiction? *Justify* your answer.

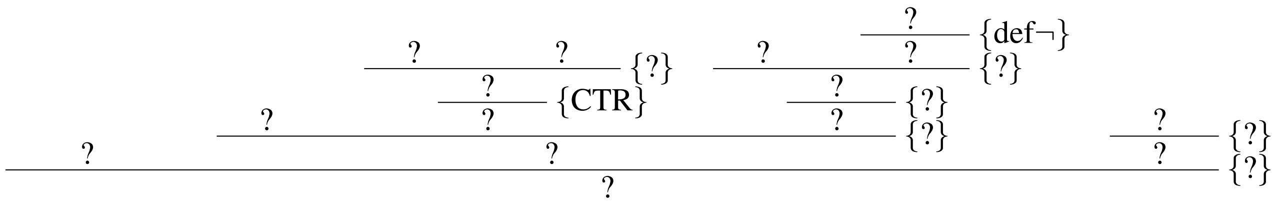
B. Is it tautologous? *Justify* your answer.

C. Draw its circuit diagram.

Question E1-3: Natural Deduction (10 points)

Consider the following partial proof of this theorem: $(p \vee q) \vee r, \neg p, \neg q \quad \vdash \quad r$

Partial proof:



Rewrite the partial proof in the space below, filling in the missing parts (marked with '?' in the proof). If there are any assumptions to be discharged, mark those assumptions and indicate the rule citations that cause those discharges.

Question E1-4: Natural Deduction (15 points)

Prove the following theorem using natural deduction: $a \to b,\ b \to c,\ c \to d \quad \vdash \quad a \to d$

For this proof, do not use the Implication Chain Rule $\{\to\text{Chain}\}$.

If there are any assumptions to be discharged, mark those assumptions and indicate the rule citations that cause those discharges.

Question E1-5: Natural Deduction (15 points)

Prove the following theorem using natural deduction: $a \to (b \vee c), \neg b, \neg c \quad \vdash \quad \neg a$

If there are any assumptions to be discharged, mark those assumptions and indicate the rule citations that cause those discharges.

Question E1-6: Equational Reasoning (20 points)

Prove the following equation using the equations of Boolean algebra:

$$((p \land q) \land r) \to s \quad = \quad (\neg s \to \neg p) \lor (\neg s \to \neg q) \lor (\neg s \to \neg r)$$

Question E1-7: Equational Reasoning (20 points)

Prove the following equation using the equations of Boolean algebra:

$$\neg p \quad = \quad (p \rightarrow q) \wedge (p \rightarrow \neg q)$$

Question E2-1: Quantified Natural Deduction (10 points)

Prove the following theorem using natural deduction:

$F(y) \vee (G(y) \vee H(y)), \forall x. \neg F(x), \forall x. \neg H(x) \quad \vdash \quad G(y)$

Question E2-2: Quantified Equational Reasoning (10 points)

Prove the following equation using equational reasoning:

$$\forall x.((A(x) \land B(x)) \to C(x)) \quad = \quad \neg\exists y.(\neg C(y) \land (A(y) \land B(y)))$$

Question E2-3: Sets (10 points)

Prove the following theorem involving the set S: $S = \varnothing \quad \vdash \quad \neg\exists x.\ x \in S$

For this proof, you may use the following rules:

$$\frac{x \in \varnothing}{\text{False}}\ \{\in\varnothing\}$$

Nothing in Empty

$$\frac{\forall x. \neg F(x)}{\neg\exists x.\ F(x)}\ \{\text{deM}\exists_F\}$$

DeMorgan Existential Forward

Question E2-4:

Circuit Minimization using Karnaugh Maps (10 points)

Consider the following Boolean function F.

```
a b c d F(a,b,c,d)  minterm
0 0 0 0     0
0 0 0 1     x
0 0 1 0     0
0 0 1 1     1
0 1 0 0     1
0 1 0 1     x
0 1 1 0     0
0 1 1 1     1
1 0 0 0     x
1 0 0 1     0
1 0 1 0     1
1 0 1 1     1
1 1 0 0     1
1 1 0 1     0
1 1 1 0     1
1 1 1 1     0
```

A. In the space in the `minterm` column above, write the minterms of F as given by the function definition above.

B. Create a Karnaugh map for F.

C. Use the Karnaugh map to find a minimum sum-of-products representation of F, give that representation, and show how you arrived at the representation from the map.

Question E2-5: Induction (20 points)

Theorem {length=n}: $\forall$n.((length [x_1, x_2, ... x_n]) = n)
Prove Theorem {length=n} using induction.

Question E2-6: Induction Redux (20 points)

Consider the following type definition and axioms:

mult::(Natural n, Num a) $\Rightarrow$ n $\rightarrow$ a $\rightarrow$ a
mult 0 a = 0 $\{\text{mult}_0\}$
mult (n+1) a = a + (mult n a) $\{\text{mult}_{n+1}\}$

Prove: $\forall$n.((x $>$ y) $\rightarrow$ ((mult n x) $\geq$ (mult n y)))

Question E2-7: More Induction Redux (20 points)

Consider the following type definition and axioms:

```
prod::(Num a) ⇒ [a] → a
prod [] = 1                       {prod[]}
prod (x:xs) = x * (prod xs)       {prod:}
```

Prove: $\forall$n.((foldr (*) 1 [x_1, x_2, ... x_n]) = (prod [x_1, x_2, ... x_n]))

Question E3-1: Proofs on Sequences (10 points)

Theorem {head++}: $\forall$n.([x_1] ++ [x_2, x_3,... x_n] = [x_1, x_2, x_3 ... x_n])
Prove Theorem {head++}.

Question E3-2: Induction (20 points)

Consider the following type definition and axioms about geometric progressions:

gp:: (Natural n, Num a) $\Rightarrow$ a $\to$ a $\to$ n $\to$ a
gp r s 0 = s $\{\text{gp}_0\}$
gp r s (n+1) = r * (gp r s n) $\{\text{gp}_{n+1}\}$

Prove Theorem $\{$geom prog$\}$: $\forall$n.(gp r s n) = s * r^n

Question E3-3: Induction on Sequences (30 points)

Consider the following type definition and axioms:

take:: Int a $\Rightarrow$ a $\rightarrow$ [b] $\rightarrow$ [b]
take n [] = [] $\{$take[]$\}$
take 0 xs = [] $\{\text{take}_0\}$
take (n+1) (x:xs) = x:(take n xs) $\{\text{take}_{n+1}\}$

Prove: $\forall$n.((length (take n as)) $\leq$ (length as))

Question E3-4: Computation Time in Numerical Systems (20 points)

Prove by induction using our computational model that for all sequences xs and all natural numbers n, (take n xs) terminates.

Question E3-5: Strong Induction with Implication (40 points)

Consider the following type definition and axioms related to the head and tail (hat) of a sequence:

hat:: [a] $\rightarrow$ ([a], [a])
hat [] = ([], []) $\{$hat[]$\}$
hat ([x_1] ++ []) = ([x_1], []) $\{$hat ++$\}$
hat ([x_1] ++ [x_2, ... x_n] ++ [x_{n+1}]) = ([x_1]++hs, ts++[x_{n+1}]) $\{$hat ++ ++$\}$
    where (hs, ts) = hat [x_2, ... x_n]

Prove: $\forall$n.((hat[x_1, x_2, ... x_n] = (as, bs)) $\rightarrow$ ((length as) $\leq$ (length [x_1, x_2, ... x_n])))

For this proof you may use Theorem $\{$head++$\}$ from Question E3-1, even if you have not proven it, and Theorem $\{$++tail$\}$ from Question E2-5 on the 2011 Final Exam, even if you have not proven it. Those theorems are restated here for your convenience:

Theorem $\{$head++$\}$: $\forall$n.([x_1] ++ [x_2, x_3,... x_n] = [x_1, x_2, x_3 ... x_n])
Theorem $\{$++tail$\}$: $\forall$n.([x_1, x_2, ... x_n] ++ [x_{n+1}] = [x_1, x_2, ... x_n, x_{n+1}])

Hint: Work out what the antecedent tells us about 'as' for each case, then substitute that into the consequent in order to prove the inequality in the consequent when the antecedent is true.

(Additional space to complete Question E3-5.)

Question E3-6: Tree Induction (30 points)

Prove: $\forall$h.$\forall$s::SearchTree.((((height s)=h) $\wedge$ ((size s) = (2^h-1)) $\wedge$ (balanced s) $\wedge$ ($\forall$x$\in$s.k$<$x)) $\rightarrow$ ((height (k,d)^:s) = (h+1)))

Hint: Do the induction on h focusing on the consequent of the implication with s being an arbitrary SearchTree with the properties given in the antecedent.

(Additional space to complete Question E3-6.)

(Extra space if needed for any Question.)