SENSITIVITY INVARIANTS FOR ACTIVE LUMPED/DISTRIBUTED NETWORKS

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The invariant nature of the sum of the sensitivities of the network functions of a general linear network is established. Results of Holt and Fidler on summed sensitivities are extended to networks containing tapered lines, v.v.t.s and c.n.i.c.s simultaneously.

In this letter, we establish the invariant nature of the sum of the sensitivities of the network functions of a general linear active lumped/distributed network. We also extend the results of Holt and Fidler on summed sensitivity invariants.

Consider a network N, consisting of lumped capacitors \( (C_i = 1/\delta_i) \), inductors \( (L_i = 1/\gamma_i) \), resistors \( (R_i = 1/G_i) \), gyrators, controlled sources, negative-impedance converters and RC and LC tapered lines. Let \( N^* \) denote the adjoint of \( N \). Then we have

\[
\Sigma V_e \phi = \Sigma V_p \phi V_p \quad \ldots (1)
\]

where \( V_e(\phi) \) and \( V_p(\phi) \) are, respectively, the voltage across (currents through) interior elements (or ports of interior elements) and external ports for \( N \) and \( N^* \). If \( Z_{ij} \) is \( i \)th element of the impedance matrix \( Z \) of \( N \), from the results of Reference 1, it can be shown that

\[
\frac{\delta Z_{ij}}{\delta \phi_i} = \phi \frac{\delta Z_{ij}}{\delta \phi_j} \quad \ldots (2)
\]

and

\[
\frac{\delta Z_{ij}}{\delta \phi_j} = -\phi \frac{\delta Z_{ij}}{\delta \phi_i} \quad \ldots (3)
\]

where \( \phi(\phi) \) and \( \phi(\phi) \) are the column vectors of currents through and voltages across the element(s) in \( N^*(N) \), respectively, and are evaluated with the \( i \)th \( (j \)th \) port of \( N^*(N) \) excited with unit current, all other ports being open-circuited.

If \( p_k \) is identified with the value of a resistor, an inductor, an inverse capacitance or a transfer resistance of a v.c.t. \( (r_i = 1/g_i) \), it is seen that

\[
p_k \frac{\delta Z_{ij}}{\delta \phi} = \phi \frac{\delta Z_{ij}}{\delta \phi} \quad \ldots (4)
\]

For a v.c.t., \( Z_{ij} \) does not exist but \( Y_{ij} \) does. It is therefore easily seen that

\[
\delta_i \frac{\delta Y_{ij}}{\delta \phi} = Y_{ij} \quad \ldots (5)
\]

where \( \delta_i \) = \( 1/\gamma_i \) is the transfer conductance of the v.c.t. Hence, from eqn. 3,

\[
\delta_i \frac{\delta Z_{ij}}{\delta \phi} = -\phi \frac{\delta Z_{ij}}{\delta \phi} \quad \ldots (6)
\]

For the gyrator, \( Z_{ij} = Z_{ji} = 0 \), \( Z_{ij} = a_{11} \) and \( Z_{ji} = a_{22} \), and it can be shown, using eqn. 2, that

\[
a_{11} \frac{\delta Z_{ij}}{\delta a_{11}} + a_{22} \frac{\delta Z_{ij}}{\delta a_{22}} = \phi \frac{\delta Z_{ij}}{\delta a_{11}} \quad \ldots (7)
\]

For an RC tapered line with resistance \( R = R_{oi} f(x) \) and capacitance \( C = C_{oi} g(x) \), it is known that

\[
Z_e = R_{oi} F \quad \ldots (8)
\]

where each element of the \( 2 \times 2 \) matrix \( F \) is a function of the product

\[
\left(sR_{oi} C_{oi} = sR_{oi} \frac{1}{D_{oi}} \right)
\]

It can be easily shown that

\[
R_{oi} \frac{\delta Z_e}{\delta R_{oi}} + D_{oi} \frac{\delta Z_e}{\delta D_{oi}} = \phi \frac{\delta Z_e}{\delta R_{oi}} \quad \ldots (9)
\]

It can also be shown that

\[
Z_{RC} \frac{\delta Z_e}{\delta Z_{RC}} = \phi \frac{\delta Z_e}{\delta Z_{RC}} \quad \ldots (10)
\]

where \( Z_{RC} = \sqrt{(R_{oi} C_{oi})} \) is the characteristic impedance of a tapered RC line.

Similarly, for an LC tapered line with inductance \( L = 1/f(x) \),

\[
L_{oi} \frac{\delta Z_e}{\delta L_{oi}} + D_{oi} \frac{\delta Z_e}{\delta D_{oi}} = \phi \frac{\delta Z_e}{\delta L_{oi}} \quad \ldots (11)
\]

and

\[
Z_{LC} \frac{\delta Z_e}{\delta Z_{LC}} = \phi \frac{\delta Z_e}{\delta Z_{LC}} \quad \ldots (12)
\]

where \( Z_{LC} = \sqrt{(L_{oi} C_{oi})} \) is the characteristic impedance of a lossless tapered line.

It can be shown that, for v.v.t.s, c.c.t.s, v.n.i.c.s and c.n.i.c.s,

\[
\phi \frac{\delta Z_e}{\delta Z_{LC}} = 0 \quad \ldots (13)
\]

If \( p_k \) is a member of the set

\[
(p_k) = (R_{oi}, L_{oi}, D_{oi}(a_{11}, a_{22}), (R_{oi} D_{oi}), Z_{LC}, r_i, \gamma_i)
\]

where the subset \( (R_{oi}, D_{oi}) \) and \( Z_{LC} \) may be replaced by \( Z_{RC} \) and \( (L_{oi}, D_{oi}) \), respectively, then for \( N^* \), using eqn. 1, we obtain

\[
\sum p_k \frac{\delta Z_{ij}}{\delta p_k} = 1 \quad \ldots (14)
\]

i.e. the sum of sensitivities of \( Z_{ij} \) is a constant independent of \( Z_{ij} \). Similarly, it can be shown that

\[
\sum p_k \frac{\delta Y_{ij}}{\delta p_k} = -1 \quad \ldots (15)
\]

\[
\sum p_k \frac{\delta Z}{\delta p_k} = 0 \quad \ldots (16)
\]

and

\[
\sum p_k \frac{\delta S}{\delta p_k} = \frac{1}{T} \quad \ldots (17)
\]

where \( T \) is a voltage or current transfer function, and \( T \) and \( S \) are unit and scattering matrices, respectively. Eqn. 17 is obtained using the procedure followed in Reference 3. If normalising resistors \( (p_0) \) are also included in \( (p_k) \) to obtain the set \( F = (p_k) \), it can be shown that the sum of the sensitivities of any scattering parameter or the group delay over the set \( F \) is zero.

Since it is more usual to calculate sensitivity with respect to \( C_i \) rather than \( D_i \), Holt and Fidler* have defined summed sensitivity for a class of active lumped networks. We now extend these results to networks containing v.c.t.s and c.n.i.c.s as well as tapered RC and LC lines. Consider a network \( N \), consisting of lumped resistors, capacitors, gyrators, RC tapered lines, controlled sources and n.i.c.s. Let the set \( G = \{a_{11}, a_{12}, (R_{oi}, D_{oi}), (L_{oi}, D_{oi}), Z_{LC}, r_i, \gamma_i\} \) and \( \{a_{11}, a_{12}\} = (C_{oi}, C_{oi}) \). Then the summed sensitivity of \( Z_{ij} \) over the set \( G \) is defined as

\[
\sum a_{ij} \frac{\delta Z_{ij}}{\delta a_{ij}} = \sum a_{ij} \frac{\delta Z_{ij}}{\delta a_{ij}} + \sum a_{ij} \frac{\delta Z_{ij}}{\delta a_{ij}} \quad \ldots (18)
\]

It should be noted that the sensitivity sum considered in eqn. 14 is

\[
\sum a_{ij} \frac{\delta Z_{ij}}{\delta a_{ij}} = \sum a_{ij} \frac{\delta Z_{ij}}{\delta a_{ij}} - \sum a_{ij} \frac{\delta Z_{ij}}{\delta a_{ij}} \quad \ldots (19)
\]

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If \( q_k \) is identified with a lumped capacitor \( C_k \),
\[
q_k \frac{\partial Z_k}{\partial q_k} = -\phi_k^* Z_k I_k \quad \text{if} \quad q_k \in Z_k \quad \ldots \quad (20)
\]
For an RC tapered line, using eqn. 8,
\[
C_{01} \frac{\partial Z_k}{\partial C_{01}} = sR_{01}^2 C_{01} F \quad \ldots \quad (21)
\]
where the dot denotes differentiation with respect to \( sR_{01} C_{01} \). Thus, from eqns. 20 and 21,
\[
\sum q_k \frac{\partial Z_k}{\partial q_k} = -\sum \phi_k^* Z_k I_k + \sum \phi_k^* sR_{01}^2 C_{01} F \quad (22)
\]
where the sum \( \sum (\phi_k) \) consists of all terms due to lumped capacitors (capacitances of RC tapered lines).

If a complex frequency variable \( s \) is treated as a parameter, it follows from the results of Reference 1 that
\[
s \frac{\partial Z_k}{\partial s} = \sum \phi_k^* s \frac{\partial Z_k}{\partial I_k} \quad \ldots \quad (23)
\]
The contribution to the right-hand side of eqn. 23 due to all resistors, gyrators, c.w.t.s and v.c.t.s is zero. The contribution from lumped capacitors is
\[
-\sum \phi_k^* Z_k I_k \quad \ldots \quad (24)
\]
while, for RC lines, from eqn. 8,
\[
s \frac{\partial Z_k}{\partial s} = sR_{01}^2 C_{01} F \quad \ldots \quad (25)
\]
Thus, from eqns. 22, 23, 24 and 25, we obtain
\[
\sum q_k \frac{\partial Z_k}{\partial q_k} = s \frac{\partial Z_k}{\partial s} \quad \ldots \quad (26)
\]
Thus, from eqns. 14, 19 and 26,
\[
\sum \frac{q_k \partial Z_k}{q_k \partial q_k} + \sum \frac{q_k \partial Z_k}{q_k \partial q_k} = 2s \frac{\partial Z_k}{Z_k} + 1 \quad (27)
\]
Hence the summed sensitivity of \( Z_k \) over \( \{q_k\} \) is an invariant and is independent of the realisation of \( Z_k \).

Similarly, it can be shown that
\[
\sum q_k \frac{\partial Y_k}{q_k \partial q_k} = 2s \frac{\partial Y_k}{Y_k} - 1 \quad \ldots \quad (28)
\]
\[
\sum q_k \frac{\partial T}{q_k \partial q_k} = 2s \frac{\partial T}{T} \quad \ldots \quad (29)
\]
Next, consider a network \( N \), consisting of lumped capacitors, inductors, lossless tapered lines, v.c.t.s, c.w.t.s and n.i.c.s. Let the set \( M = \{m_i\} \) be \( \{L_i, C, \{L_{01}, C_{01}\}\} \). Then the summed sensitivity of a network function over the set \( M \) is given by
\[
\sum m_i \frac{\partial F}{m_i \partial m_i} = s \frac{\partial F}{\partial s} \quad \ldots \quad (30)
\]
where \( F \) can be an impedance parameter, an admittance parameter or a voltage or current transfer function.

The basis-free normalised scattering matrix \( S \) for a linear time-invariant network with a resistive reference is given by\(^2\)
\[
S = U - 2p^4(Z + p) - p^4 \quad \ldots \quad (31)
\]
For network \( N \), the summed sensitivity of \( S \) over the set \( \{q_k\} \) could be obtained by using eqns. 14 and 31 as
\[
\sum q_k \frac{\partial S}{q_k \partial q_k} = 2s \frac{\partial S}{\partial s} + 4(U - S^2) \quad \ldots \quad (32)
\]
If the set \( \{\rho_i\} \) of normalising resistors is also included,