Analysis and Synthesis of the $K$- and $Y$-Matrices of Resistive $n$-Port Networks

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With 3 Figures (Received 25th May 1973)

I. Introduction

The potential factor matrix $K$ of an $n$-port network was first introduced in connection with establishing a criterion for the proper parallel connection of $n$-port networks [1]. An extensive use of the concept of potential factors was later made in the realisation of a real symmetric dominant matrix as the $Y$-matrix of an $n$-port network [2, 3]. Certain aspects of the relationship between the modified cutset matrix and the potential factors of an $n$-port network were dealt with in [4]. Recently Lempel and Cederbaum have discussed the synthesis of $K$-matrices\(^2\) of resistive $n$-port networks [5]. In a more recent paper [6], the usefulness of the concept of potential factors in the realisation of $Y$-matrices of $n$-port networks and the synthesis of $K$-matrices of $(n + 2)$-node resistive $n$-port networks have been discussed.

In this paper analysis and synthesis of $K$- and $Y$-matrices of resistive $n$-port networks are considered. In Section II, an equation relating the modified cutset matrix and the $K$-matrix of an $n$-port network and certain results regarding port-vertex equivalent $n$-port networks are given. A procedure is given in Section III for the generation of padding $n$-port networks. Synthesis of $K$ and $Y$ matrices is discussed in Section IV. A lower bound on the number of conductances required for the realisation of $Y$-matrices of $n$-port networks having a prescribed port configuration is also obtained in Section IV.

Unless stated otherwise we follow the notation used in [6].

II. Relationship between the modified cutset Matrix and the $K$-Matrix of an $n$-Port Network

We consider a resistive $n$-port network $N$ having a port configuration $T$. We assume, without loss of generality, that $N$ contains no internal vertices. The linear graph of $N$ will be denoted by $G$. Let the port configuration $T$ be in $p$ ports $T_i$, $i = 1, 2, ..., p$. Let $T_0$ be a tree of $G$ and let $T$ be a subgraph of $T_0$. The edges of $T$ will be referred to as port branches and the remaining edges of $T_0$ will be called non-port branches (n. p. b.). The $j^{th}$ port of $N$ and the corresponding port branch will be both denoted by $P_i$. The set of branches of $T_i$ and the corresponding set of ports will be both denoted by $P_i$.

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2 In the definition used by Lempel and Cederbaum, all the diagonal entries of the $K$-matrix are equal to zero.
Let $C_0$ be the fundamental cutset matrix of $G$ with respect to $T_0$. Let $C_1(C_2)$ be the submatrix of $C_0$ such that the rows of $C_1(C_2)$ correspond to port (non-port) branches. If $V_e$, $V_p$, and $V_n$ denote the column matrices of edge voltages, port voltages and non-port branch voltages, then

$$V_n = M^t V_p$$
$$= [m_{ij}]^t V_p \quad (1)$$

where $m_{ij}$ is the voltage across the $j$th non-port branch when port $i$ is excited with a source of unit voltage and all the other ports are short-circuited, and

$$V_n = C^t V_p \quad (2)$$

where $C$ is the modified cutset matrix of $N$ and is given by [7]

$$C = C_1 + MC_2. \quad (3)$$

We now proceed to obtain an equation relating the matrix $M$ to the potential factor matrix $K$.

We first define an $(n \times p)$ matrix

$$\bar{K} = [\bar{K}_{ij}] = \begin{bmatrix} \bar{K}_1 \\ \bar{K}_2 \\ \vdots \\ \bar{K}_p \end{bmatrix} \quad (4)$$

as follows:

a) $i$th row of $\bar{K}$ corresponds to port $p_i$;

b) $i$th column of $\bar{K}$ corresponds to the set of ports $P_i$;

c) the rows of the submatrix $\bar{K}_i$ corresponds to the ports of $P_i$;

d) if $j \neq i$, then the $j$th column of $\bar{K}_i$ is equal to some column of $K_{ij}$;

e) if $j = i$, then the $j$th column of $\bar{K}_i$ consists of 1's only.

From the above definition of $\bar{K}$, we observe that if $p_i \in P_j$, then $\bar{K}_{ij}$ represents the voltage of $P_j$ with respect to the negative reference terminal of port $p_i$, when $p_i$ is excited with a source of unit voltage and all the other ports are short-circuited. Also, the port configuration $T$ and $\bar{K}$ completely specify $K$.

Let $\bar{T}$ be the linear graph obtained after short-circuiting all the port branches of $T_0$. $\bar{T}$ will have $p$ vertices, $v_i$, $i = 1, 2, \ldots, p$, the vertex $v_i$ corresponding to the set of ports $P_i$. The $(p - 1)$ non-port branches of $T_0$ will form the edges of $\bar{T}$. Let $A$ be the incidence matrix of $\bar{T}$, the $i$th row of $A$ corresponding to $v_i$ and the $j$th column corresponding to the $j$th non-port branch.

Let $\bar{T}_i$ be the graph obtained from $T_0$ after short-circuiting all the port branches except $p_i$. If $p_i \in P_j$, then the $(p - 1)$ vertices $v_i$, $i = 1, 2, \ldots, p$, $i \neq j$ and the two vertices of $p_i$ will constitute the vertex set of $\bar{T}_i$.

Following are the possible ways in which the $r$th non-port branch (rth n. p. b.) can be situated in $\bar{T}_i$, with respect to $p_i$.

a) rth n. p. b. is not incident at the vertices of $p_i$ (Fig. 1a).

b) rth n. p. b. is incident at and oriented towards the positive reference terminal of $p_i$ (Fig. 1b).

c) rth n. p. b. is incident at and oriented away from the positive reference terminal of $p_i$ (Fig. 1c).
d) $r^{th}$ n. p. b. is incident at and oriented towards the negative reference terminal of $p_i$ (Fig. 1d).

e) $r^{th}$ n. p. b. is incident at and oriented away from the negative reference terminal of $p_i$ (Fig. 1e).

![Diagram](image)

We next define an $(n \times p - 1)$ matrix $\tilde{A} = [\tilde{a}_{ir}]$ as follows:

a) $i^{th}$ row of $\tilde{A}$ corresponds to $p_i$ and the $j^{th}$ column corresponds to the $j^{th}$ n. p. b.

b) $\tilde{a}_{ir} = 0$, if in $T_i$, either (i) the $i^{th}$ n. p. b. is not incident at the vertices of $p_i$, or (ii) $r^{th}$ n. p. b. is incident at the positive reference terminal of $p_i$.

c) $\tilde{a}_{ir} = 1$, if in $T_i$, $r^{th}$ n. p. b. is incident at and oriented towards the negative reference terminal of $p_i$.

d) $\tilde{a}_{ir} = -1$, if in $T_i$, $r^{th}$ n. p. b. is incident at and oriented away from the negative reference terminal of $p_i$.

We then have the following theorem.

**Theorem 1:**

$$M = KA + \tilde{A}.$$  

**Proof:**

The $(i, r)$ entry $m_{ir}$ of $M$ represents, by definition, the voltage across the $r^{th}$ n. p. b. when port $p_i$ is excited with a source of unit voltage and all the other ports are short-circuited. We shall denote the $(i, r)$ entry of $KA$ as $(KA)_{i,r}$. We shall consider the five possible ways enumerated earlier, in which the $r^{th}$ n. p. b. can be situated in $T_i$ (Figs. 1 (a), (b), (c), (d) and (e)) and obtain in each case $m_{ir}$ ($KA)_{i,r}$ and $\tilde{a}_{ir}$.

**Case A:** In $T_i$, $r^{th}$ n. p. b. is situated with respect to $p_i$ as in Fig. 1 (a).

$$m_{ir} = k_{ir} - \tilde{k}_{ik}$$

$$(KA)_{i,r} = k_{ir} - \tilde{k}_{ik}$$

$$\tilde{a}_{ir} = 0.$$
Case B: In $\overline{T}_i$, $r$th n. p. b. is situated with respect to $p_i$ as in Fig. 1 (b).

$$m_{ir} = k_{ir} - 1$$
$$\overline{(KA)}_{ir} = k_{ir} - 1$$
$$a_{ir} = 0$$

Case C: In $\overline{T}_i$, $r$th n. p. b. is situated with respect to $p_i$ as in Fig. 1 (c).

$$m_{ir} = 1 - k_{ir}$$
$$\overline{(KA)}_{ir} = 1 - k_{ir}$$
$$a_{ir} = 0$$

Case D: In $\overline{T}_i$, $r$th n. p. b. is situated with respect to $p_i$ as in Fig. 1 (d).

$$m_{ir} = k_{ik}$$
$$\overline{(KA)}_{ir} = k_{ik} - 1$$
$$a_{ir} = 1$$

Case E: In $\overline{T}_i$, $r$th n. p. b. is situated with respect to $p_i$ as in Fig. 1 (e).

$$m_{ir} = -k_{ik}$$
$$\overline{(KA)}_{ir} = 1 - k_{ik}$$
$$a_{ir} = -1$$

We observe that in all the cases considered above

$$m_{ir} = (\overline{KA})_{ir} + a_{ir}.$$ 

Hence the theorem.

It follows from theorem (1) and equation (3) that

$$C = C_1 + (\overline{KA} + \overline{A}) C_2.$$  

We note that $\overline{K} = K$, in the case of $2n$-node $n$-port networks. Hence in that case

$$C = C_1 + (KA + \overline{A}) C_2.$$  

Eq. (5) and (6) are respectively similar to Eq. (61) and (10) of reference [5]. The latter equations involve the use of certain submatrices of the matrix relating the incidence and fundamental cutset matrices of a graph obtained from $G$.

Consider, next, an $n$-port network, $N^*$ constructed on $N$. Let $T^*$, the port configuration of $N^*$, also be in $p$ parts $T_i^*$, $i = 1, \ldots, p$, such that the vertices of $T_i^*$ are the same as those of $T_i$. The $n$-port networks $N$ and $N^*$ defined as above will be referred to as port-vertex equivalent $n$-port networks.

Let $C^*$ be the modified cutset matrix of $N^*$. Let $Y^*$, $V_p^*$ and $M^*$ be defined similarly. If

$$V_p = A^tV_p^*$$

then it is easy to show that

$$M^* = AM$$

$$C^* = AC$$
and
\[ Y^* = AYA^t. \] (10)

Further, if \( N \) is a padding \( n \)-port network, then it follows from (10) that \( N^* \) is also a padding \( n \)-port network.

III. Synthesis of padding \( n \)-Port Networks

We obtain, in this section, a procedure for the generation of padding \( n \)-port networks, having specified potential factors and a prescribed port configuration.

We shall assume, without loss of generality, that each connected part \( T_i \), \( i = 1, 2, \ldots, p \) of the port configuration of the required padding \( n \)-port network \( N \) is a lagrangian tree. The set of vertices of \( T_i \) will be denoted as \( i_0, i_1, i_2, \ldots, i_{n_i} \), with \( i_0 \) as the star vertex of \( T_i \). The \( m \)th port of \( P_i \) will be denoted by \( P_{i(m)} \), \( i_m \) and \( i_0 \) are the positive and negative reference terminals of \( P_{i(m)} \). \( T_i \) and the polarities of \( P_{i(m)} \) are as shown in Fig. 2.

\[ \text{Fig. 2} \]

The conductance of the edge connecting the vertices \( i_k \) and \( i_m \) of \( N \) will be denoted by \( g_{ik,m} \). We further define \( S_{ikj} \) and \( S_{ij} \) as follows:

\[ S_{ikj} = \sum_{m=0}^{n_j} g_{ik,m}, j \neq i \] (5)

case

\[ S_{ij} = \sum_{k=0}^{n_k} S_{ikj}, j \neq i \] (6)

The the

\[ = \sum_{m=0}^{n_i} S_{jm,i^*} \]

The network obtained from \( N \) after short circuiting all the ports will be denoted by \( \overline{N} \), and the network obtained after short-circuiting all the ports except \( P_{i(m)} \) and connecting a source of unit voltage across \( P_{i(m)} \) will be denoted by \( N_{i(m)} \). We observe that (i) the \( p \) vertices \( v_{i_0}, i = 1, 2, \ldots, p \) will constitute the vertex set of \( \overline{N} \). (ii) the \( (p - 1) \) vertices \( v_{r}, r = 1, 2, \ldots, p, r \neq i \) and the vertices of \( P_{i(m)} \) will constitute the vertex set of \( N_{i(m)} \) and (iii) \( C_2 \) is the fundamental cutset matrix of \( N \) with respect to \( T \). If \( A_i \) is the reduced incidence matrix of \( \overline{N} \) with \( v_{i_0} \) as the reference vertex, then \( C_2 \) can be expressed as

\[ C_2 = R_i A_i. \] (7)

It is well known that \( R_i \) is non-singular. Further we denote by \( k_{ij}, j \neq i \), the voltage of the set of ports \( P_i \) when \( P_{i(k)} \) is excited with a source of unit voltage
and all the other ports are short circuited. We note that $k_{i(\ell),j}$ is equal to some element of the $k^{th}$ row of $K_i$, $j \neq i$.

Given the port configuration $T$ and the potential factors, the modified cutset matrix $C$ of the required padding $n$-port network $N$ can be easily constructed. It has been shown that a real diagonal matrix $C$ will represent the edge conductance matrix of a padding $n$-port network $N$ if and only if the following equations are satisfied [4].

\[
CGC_2^k = 0 \quad (12a)
\]

\[
CGC_1^k = 0 \quad (12b)
\]

and

\[
\det (C_2^k G C_2^k) = 0. \quad (12c)
\]

We next proceed to solve Eq. (12) for $G$.

Consider first Eq. (12a). This equation can be written as a set of linear equations with $S_i v_j$'s as unknowns. If $C_i^{(k)}$ denotes the row of $C$ corresponding to $P_i^{(k)}$, then equation (12a) can be written as

\[
C_i^{(k)} G C_2^k = 0, \quad i = 1, 2, \ldots, p,
\]

\[
k = 1, 2, \ldots, n_i. \quad (13)
\]

From (11) and (13) we get

\[
C_i^{(k)} G A_1^k = 0, \quad i = 1, 2, \ldots, p,
\]

\[
k = 1, 2, \ldots, n_i. \quad (14)
\]

We note that Eq. (13) and (14) are equivalent since $R_i$ is non-singular. The equation

\[
C_i^{(k)} G A_1^k = 0 \quad \text{for some } i \text{ and some } k
\]

represents the following set of $(p - 1)$ equations.

\[
S_{k,j} (k_{i(\ell),j} - 1) + \sum_{m=0}^{n_i} S_{i,m} k_{i(\ell),j}
\]

\[+ \sum_{m=1}^{p} S_{j,m} (k_{i(\ell),j} - k_{i(\ell),m}) = 0
\]

\[j = 1, 2, \ldots, p, \quad j \neq i. \quad (15)
\]

Eq. (15) can be easily identified as Kirchhoff's current law equation for $N_i^{(k)}$ at the $(p - 1)$ vertices $v_r$, $r = 1, 2, \ldots, p$, $r \neq i$. Solving (15) for $S_{i,j}$ and generalising the result we get

\[
S_{i,j} = \sum_{m=0}^{n_i} S_{i,m} k_{i(\ell),j} + \sum_{m=1}^{p} S_{j,m} (k_{i(\ell),j} - k_{i(\ell),m})
\]

\[i = 1, 2, \ldots, p,
\]

\[k = 1, 2, \ldots, n_i. \quad (16)
\]

and

\[j = 1, 2, \ldots, p, \quad j \neq i.
\]

Since

\[
S_{i,j} = \sum_{k=0}^{n_i} S_{k,j}.
\]
We get from (16)

\[ S_{i,j} = S_{ij} - \sum_{k=1}^{n} S_{kj} \]

\[ = S_{ij} \left( 1 - \sum_{k=1}^{n} k_{i(k),j} \right) - \sum_{k=1}^{n} \sum_{m=1}^{p} S_{j,m} (k_{i(k),j} - k_{i(k),m}) \]  \( i = 1, 2, \ldots, p \) and \( j = 1, 2, \ldots, p \), \( j \neq i \).  \( (17) \)

Eq. (16) and (17) can be used to evaluate all \( S_{i,j} \)'s after assuming arbitrary values for \( S_{ij} \)'s. \( S_{ij} \)'s so obtained will satisfy Eq. (12a).

We next proceed to solve Equation (12b) and obtain expressions for the edge conductances of \( N \) in terms of \( S_{ij} \)'s.

Let \( C_{i,j(m)} \) denote the row of \( \mathbf{C}_i \) corresponding to \( P_{j(m)} \). Then, taking into account the symmetry of the short-circuit conductance matrix \( \mathbf{CG}_1 \), the following sets of equations are obtained from Eq. (12b).

\[ C_{i}^{(k)} \mathbf{GC}_{i,j(m)}^{4} = 0, \quad i = 1, 2, \ldots, p - 1, \]

\[ k = 1, 2, \ldots, n_0, \]

\[ j = 2, 3, \ldots, p, \quad j > i, \quad m = 1, 2, \ldots, n_j. \]  \( (18a) \)

\[ C_{i}^{(k)} \mathbf{GC}_{i,i(m)}^{4} = 0, \quad i = 1, 2, \ldots, p, \]

\[ k = 1, 2, \ldots, n_i - 1, \]

\[ m = 2, 3, \ldots, n_i, \quad m > k. \]  \( (18b) \)

and

\[ C_{i}^{(k)} \mathbf{GC}_{i,i(k)}^{4} = 0, \quad i = 1, 2, \ldots, p, \]

\[ k = 1, 2, \ldots, n_i. \]  \( (18c) \)

Consider the equation

\[ C_{i}^{(k)} \mathbf{GC}_{i,j(m)}^{4} = 0, \quad \text{for some } i, \ k, \ j > i, \ \text{and } m. \]

This equation can be written as

\[ g_{i,k}^{j,m}(k_{i(k),j} - 1) + \sum_{r=0}^{n_i} g_{i,r}^{j,m} k_{i(k),j} + \sum_{r=1}^{p} S_{j,m}^{r}(k_{i(k),j} - k_{i(k),r}) = 0. \]  \( (19) \)

Solving (19) for \( g_{i,k}^{j,m} \) and generalising the result, we get

\[ g_{i,k}^{j,m} = S_{i,m}^{k} k_{i(k),j} + \sum_{r=1}^{n_i} S_{j,m}^{r}(k_{i(k),j} - k_{i(k),r}) \]

\[ i = 1, 2, \ldots, p - 1 \]
\[ k = 1, 2, \ldots, n_i \]
\[ j = 2, 3, \ldots, p, \quad j > i \]
\[ m = 1, 2, \ldots, n_j. \]  \( (20) \)
Values for $g_{ikm}$’s obtained by using (20) will satisfy (18a). Further $g_{ikm}$ and $g_{ikj}$ and $g_{iks}$ can be obtained as

$$g_{ikm} = S_{ikm} - \sum_{k=1}^{n_i} g_{ikm}$$

and

$$g_{ikj} = s_{ikj} - \sum_{m=1}^{n_j} g_{ikjm}$$

and

$$g_{iks} = S_{iks} - \sum_{m=1}^{n_i} g_{iksm}$$

\[i = 1, 2, \ldots, p - 1\]
\[k = 1, 2, \ldots, n_i\]
\[j = 2, 3, \ldots, p, j > i\]
\[m = 1, 2, \ldots, n_j.\]

Eq. (20) and (21) will enable us to obtain conductances of edges connecting vertices in different $T_c$’s.

We then consider equation (18b). The equation

$$C_{ik}^t G_{ik}^{t(m)} = 0$$

for some $i, k$ and $m > k$ can be written as

$$-g_{ikm} - \sum_{j=1}^{p} S_{ikj}^{t(k),j} = 0. \quad (22)$$

Solving Eq. (22) for $g_{ikm}$ and generalising the result we get

$$g_{ikm} = \sum_{j=1}^{p} S_{ikj}^{t(k),j}$$

\[i = 1, 2, \ldots, p,\]
\[k = 1, 2, \ldots, n_i - 1,\]
\[m = 2, 3, \ldots, n_j, m > k.\]

Values for $g_{ikm}$’s obtained using (23) will satisfy Eq. (18b).

Finally, we consider equation (18c). The equation

$$C_{ik}^{(l)} G_{ik}^{t(l)} = 0,$$

for some $i$ and some $k$ can be written as

$$g_{ik0} + \sum_{j=1}^{p} S_{ikj}^{l(k),j} = \sum_{m=1}^{n_i} g_{ikm} = 0 . \quad (24)$$

From Eq. (24) we get

$$g_{ik0} + \sum_{m=1}^{n_i} g_{ikm} + \sum_{j=1}^{p} (k_{ikj}^{l(k),j} - 1) S_{ikj}^{l(k),j} .$$
We get from Eq. (23) and (25)
\[
g_{ikj} = \sum_{m=1}^{n_i} \sum_{j=1}^{p} \sum_{j=1}^{p} S_{im} k_{i(k),j} - \sum_{j=1}^{p} (1 - k_{i(k),j}) S_{i,j}
\]
\[= \sum_{j=1}^{p} k_{i(k),j} S_{i,j}.
\]

The last step in Eq. (26) follows after equating to zero the sum of the currents in those edges of \(N_{(k)}\), connecting vertices in \(T_i\) to all other vertices. Generalising the result obtained in Eq. (26) we get,
\[
g_{ikj} = - \sum_{j=1}^{p} k_{i(k),j} S_{i,j},
\]
\[i = 1, 2, \ldots, p,
\]
\[k = 1, 2, \ldots, n_i.
\]

Values for \(g_{ikj}\)'s obtained using Eq. (27) will satisfy (18c).

The discussions up to this point may be summarized as follows:

a) Assuming arbitrary values for \(S_{i,j}\)'s, determine \(S_{ikj}\)'s using Eq. (16) and (17).

b) Use the values of \(S_{ikj}\)'s so obtained, in Eq. (20), (21), (23) and (27), and determine the values for the edge conductances \(g_{ikm}\)'s of \(N\).

c) The values of edge conductances so obtained will satisfy Eq. (12a) and (12b).

We next turn to (12c). Need for padding network synthesis arises in the realisation of \(K\) and \(Y\) matrices by \(n\)-port networks having no negative conductances. If \(N\) is to be the padding \(n\)-port network of some \(n\)-port network containing no negative conductances then all \(S_{ij}\)'s should be chosen nonnegative. Further if \(\overline{N}\) is connected and contains no negative conductances (i.e., all \(S_{ij}\)'s are non-negative), then \((C_0G_0C_0^*)\) will be nonsingular and (12c) will be satisfied. So, while selecting values for \(S_{ij}\)'s it must be ensured

i) all \(S_{ij}\)'s are non-negative, and

ii) some \(S_{ij}\)'s must be positive so that \(\overline{N}\) is connected.

In the foregoing, expressions for edge conductances of \(N\) have been obtained, assuming that each connected part of \(T\) is a langrangian tree. This assumption, however, involves no loss of generality, as may be seen from the following.

If any arbitrary connected port configuration \(T^*\) and the corresponding potential factors are specified then, the potential factors corresponding to \(T\), in which each connected part is a langrangian tree, can be easily obtained (Equation 8 in the previous section). If an \(n\)-port padding network \(N\) having the port configuration \(T\) and the newly determined potential factors is generated then the \(n\)-port network \(N^*\) with the port configuration \(T^*\) will also be a padding network with its potential factors as specified. It may be noted \(N\) and \(N^*\) are port-vertex equivalent \(n\)-port networks.

If, however, the port configuration \(T^*\) alone is specified, then we should first generate \(N\) assuming values for all distinct \(S_{ij}\)'s as well as for all \(k_{i(k),j}\)'s. The \(n\)-port network \(N^*\) port-vertex equivalent to \(N\) and having the prescribed port configuration \(T^*\) will be the padding \(n\)-port network required.

This completes our discussions on the synthesis of padding \(n\)-port networks having any arbitrary connected port configuration and having specified potential factors. The usefulness of these results will be discussed in the next section.
Example 1:

It is required to generate a 3-port padding network having the port configuration shown in Fig. 3a. The potential factors of the required network should be as follows:

\[ k_{1(1,2)} = k_{12} = 0.5; \quad k_{1(1,3)} = k_{13} = 0.6 \]
\[ k_{2(1,1)} = k_{21} = 0.4; \quad k_{2(1,3)} = k_{23} = 0.5 \]
\[ k_{3(1,1)} = k_{31} = 0.3; \quad k_{3(1,2)} = k_{32} = 0.2. \]

Assume \( S_{ij} \)'s as follows: \( S_{12} = 10, S_{13} = 20; S_{23} = 10. \) Using Eq. (16) and (17) \( S_{ij} \)'s are obtained as

\[ S_{1,2} = 4; \quad S_{1,2} = 6; \quad S_{1,3} = 13; \quad S_{1,3} = 7; \]
\[ S_{2,1} = 2; \quad S_{2,1} = 8; \quad S_{2,3} = 7; \quad S_{2,3} = 3; \]
\[ S_{3,1} = 7; \quad S_{3,1} = 13; \quad S_{3,2} = 1; \quad S_{3,2} = 9. \]

For example,

\[ S_{3,2} = S_{23} k_{32} + S_{21}(k_{32} - k_{31}) \]
\[ = 10 \cdot 0.2 + 10 \cdot (-0.1) = 1 \]

and

\[ S_{3,2} = S_{23} - S_{3,2} = 9. \]
Using the values of $S_{i,j}$'s so obtained in Eq. (20), (21), (23) and (27) we can get the edge conductances of the required 3-port network. For example

\[
g_{1,2} = S_{2,1}k_{12} + S_{2,3}(k_{12} - k_{13})
\]
\[
= 2 \cdot 0.5 - 7 \cdot 0.1 = 0.3
\]
\[
g_{1,3} = -S_{1,2}k_{12} - S_{1,3}k_{13}
\]
\[
= -6 \cdot 0.5 - 7 \cdot 0.6 = -7.2.
\]

The required 3-port padding network is shown in Fig. 3b.

IV. Synthesis of K- and Y-Matrices of n-Port Networks

a) Synthesis of the K-Matrix

Synthesis of the potential factor matrix $K$ of an $n$-port network requires the solution of the following two problems:

i) Determination of the port configuration $T$ appropriate to $K$.

ii) Determination of an $n$-port network containing no negative conductances and having the port configuration $T$ and the specified $K$-matrix.

In their solution of the first problem, Lempel and Cederbaum [5] first assume the port configuration to be in $n$ parts and then obtain the modified cutset matrix, appropriate to the assumed port configuration and the given $K$-matrix. This modified cutset matrix can be determined either by using Eq. (10) of reference [5] or Eq. (6) of this paper or by inspection of the assumed port configuration and the given $K$ matrix. From the modified cutset matrix so obtained, the port configuration $T$ is determined by the application of a simple procedure, which yields a unique port configuration $T$ for a given $K$ matrix.

To solve the second problem, Lempel and Cederbaum first determine the modified cutset matrix $C$, appropriate to the port configuration $T$ and the specified $K$ matrix. Then linear programming technique is applied to obtain a non-negative $G$, if one exists, satisfying the equation

\[
GCG^T = 0.
\]

In this section, we give a new necessary and sufficient condition to test the existence of a resistive $n$-port network containing no negative conductances and having a specified $K$-matrix and a port configuration $T$ appropriate to the matrix $K$. We assume, without loss of generality, that each connected part of the port configuration $T$ is a lagrangian tree.

Let the column matrices of $(S_{i,j})_p$'s and $(S_{i,j})_p$'s of an $n$-port network $N_p$ be denoted by $S_p$ and $S_p$ respectively. Eq. (16) and (17) can be together written in matrix form as

\[
S_p = PS_p
\]

where each entry of the matrix $P$ is a linear combination of some potential factors. If $(S_{i,j})_p$'s are such that $N_p$ is connected then the corresponding $S_p$ will be called non-trivial. It is shown in [6] that

i) if all $(S_{i,j})_p$'s of a padding $n$-port network $N_p$ are non-negative then a network of departure $N_d$ can be found so that the parallel combination $N$ of $N_p$ and $N_d$ contains no negative conductances;

ii) an $n$-port network and its padding network have the same $K$-matrix, and

iii) $(S_{i,j})_p = S_{i,j}$ and $(S_{i,j})_p = S_{i,j}$.

Theorem (2) then follows.
Theorem 2

Let each connected part of the port configuration $T$ appropriate to a given potential factor matrix $K$ be a lagrangian tree. The matrix $K$ can be realised by a resistive $n$-port network containing no negative conductances if and only if there exists a non-trivial $S$ such that $S \geq 0$ and $PS \geq 0$.

Following steps may then be used for the synthesis of a $K$-matrix:

i) Obtain a non-trivial value of $S$, if it exists, equal to $S_a$ such that $S_a \geq 0$ and $PS_a \geq 0$ [9, 14].

ii) Construct a $p$-$n$-port network $N_p$ having the matrix $K$ as its potential factor matrix and such that its $\overline{S}$ matrix is equal to $PS_a$.

iii) Determine a suitable $N_a$ so that the parallel combination $N$ of $N_a$ and $N_p$ contains no negative conductances.

iv) The network $N$ realizes the matrix $K$.

We now wish to draw attention to the following.

1. According to the procedure given in [6] to determine a suitable $N_a$ for a given $N_p$ in which all $(S_ik)_p$'s are non-negative

$$ (g_{ikm})_d = -(g_{ikm})_p $$

and

$$ (g_{ikm})_d = -(g_{ikm})_p + \frac{S_{ik}S_{jm}}{S_{ij}}, \quad i \neq j. $$

Since

$$ g_{ikm} = (g_{ikm})_d + (g_{ikm})_p \quad \text{for all } i \text{ and } j, $$

we get

$$ g_{ikm} = 0 $$

and

$$ g_{ikm} = \frac{S_{ik}S_{jm}}{S_{ij}} \quad \text{for all } i \text{ and } j, \quad j \neq i. $$

Thus determination of $N$ requires the evaluation of only $(g_{ikm})_p$'s using (31).

2. It can be shown using Eq. (16) that for every vertex $i_k$ there exists a $j$ such that $S_{ik}$ is non-negative if all $S_{ij}$'s are non-negative. Thus of the $(n + p) \cdot (p - 1)$ elements of the vector $\overline{S}$, $(n + p)$ elements will be non-negative if all $S_{ij}$'s are non-negative. Hence the total number of constraints involved in the solution of the linear program implied in theorem (2) is only $(n + p)(p - 2)$. In contrast the number of constraints used in the procedure given in [5] is $n(p - 1)$. It may be noted that for

$$ n > p(p - 2) $$

the procedure given in this section for $K$-matrix synthesis involves a smaller number of constraints than used by Lempel and Cederbaum [5]. Further the present procedure involves $\frac{p(p - 1)}{2}$ number of unknowns which is less than the minimum number of unknowns, namely $2p(p - 1)$ used in [5].

The new approach given in this section for $K$-matrix synthesis provides a greater insight into the nature of the $K$-matrix synthesis problem. In fact following the same approach a simple necessary and sufficient condition has already been obtained for the synthesis of the $K$-matrices of $(n + 2)$-node $n$-port networks. Further, since
this procedure essentially requires the synthesis of a suitable padding network having a specified \( K \)-matrix, it can be readily used in \( Y \)-matrix synthesis as discussed in Section IV (b).

b) Synthesis of the \( Y \)-Matrices of \( n \)-port Networks with more than \((n + 1)\) Nodes

The only approach available for the synthesis of the \( Y \)-matrices of \( n \)-port resistive networks having more than \((n + 1)\)-nodes is due to Guillemin [8]. This approach essentially requires the determination of a suitable padding \( n \)-port network \( N_p \) for a given network of departure \( N_d \). As a result a number of procedures have been proposed in the past for the generation of padding \( n \)-port networks [8, 10, 11, 12]. The procedure for padding network generation given in Section III is yet another contribution in this direction.

A significant feature of this new procedure is that all the parameters used herein namely \( S_i^j \)'s and \( k_{ij} \)'s can be readily identified with certain quantities of the padding network \( N_p \) to be realized. Also these quantities happen to be the same for both \( N_p \) and \( N \). The procedures for padding network synthesis given in [8, 10, 11], and [12] do not permit such straightforward identification for all the parameters. This feature of the new procedure is of help in the synthesis of \( Y \) matrices of \( RLC \) \( n \)-port networks. In the synthesis of such networks, it is required to realize a set of real symmetric \( Y \) matrices by resistive \( n \)-port networks, all having the same modified cutset matrix [3], [8]. If a network \( N_1 \) realizing one of these matrices is known, then all networks realizing the other matrices should have the same modified cutset matrix as \( N_1 \). This leads us to the problem of synthesis of a resistive \( n \)-port network having a prescribed \( Y \) matrix, a prescribed port configuration and specified potential factors \( k_{ij} \)'s. To solve this, we may proceed as follows. We assume that each connected part of the port configuration \( T \) is a Lagrangian tree.

Let \( \{g\}_d \) be the column matrix of edge conductances of the network of departure \( N_d \) with respect to the given \( Y \) and \( T \). Let \( \{g\}_p \) be the column matrix of edge conductances of a required padding network \( N_p \). It follows from Eq. (16), (17), (20), (21), (23) and (27) that \( \{g\}_p \) can be related to \( S \), the column matrix of \( S_i^j \)'s, as

\[
\{g\}_p = QS
\]

where each entry of \( Q \) is a function of \( k_{ij} \)'s. Hence \( Q \) can be determined from the values specified for \( k_{ij} \)'s. Since the parallel combination of \( N_p \) and \( N_d \) should contain no negative conductances, it is required that

\[
\{g\}_p = QS \geq - \{g\}_d.
\]  \hspace{1cm} (32)

If a non-negative and non-trivial value of \( S \) equal to \( S_d \) satisfying (32) exists then the column matrix \( \{g\} \) of conductances of the required \( n \)-port network will be given by

\[
\{g\} = \{g\}_d + QS_a.
\]

Thus it follows from (32) that when \( T \), \( Y \) and \( k_{ij} \)'s are specified, \( n \)-port synthesis problem simplifies to one of solving a linear program. This is in contrast to the non-linear equations involved when \( Y \) and \( T \) alone are specified. Following the approach outlined above, a simple necessary and sufficient condition has already been established for the synthesis of \((n + 2)\)-node \( n \)-port networks having prescribed \( Y \) and \( K \)-matrices [13]. The procedure for padding network synthesis given in [8, 10, 11, 12] will not be of help in the synthesis of \( Y \)-matrices of \( RLC \) \( n \)-port networks.
c) **Lower Bound on the Number of Conductances required for the Synthesis of a Y-Matrix**

We next establish a lower bound on the number of conductances required for the realisation of a real symmetric matrix \( Y \) as the short-circuit conductance matrix of a resistive \( n \)-port network containing no negative conductances and having a prescribed port configuration \( T \).

Consider a resistive \( n \)-port network \( N \) containing no negative conductances. Let each connected part of the port configuration \( T \) of \( N \) be a lagrangian tree.

For every port \( P_{(k)} \) of \( N \) there exists a \( j \) such that

\[
k_{(k),j} \geq k_{(k),r}, \quad r = 1, 2, \ldots, p, \quad r \neq i, \quad r \neq j. \tag{33}
\]

Since \( N \) contains no negative conductances all \( S_{ij} \)'s are non-negative. Then it follows from Eq. (33) and (20) that for every vertex \( i_k \), \( k \neq 0 \) there exists a \( j \) such that

\[
(g_{ijm})_p \geq 0 \quad \text{for all} \quad m = 0, 1, 2, \ldots, n_j. \tag{34}
\]

Let \( N^* \) be an \( n \)-port network port-vertex equivalent to \( N \). Let the star vertex of each \( T^*_l \) be different from that of \( T_l \). Then following the same line of argument as above, we can show that for every vertex \( i_0 \) there exists a \( j \) such that

\[
(g_{ijm})_p \geq 0 \quad \text{for all} \quad m = 0, 1, 2, \ldots, n_j. \tag{35}
\]

Since the padding networks of port-vertex equivalent \( n \)-port networks are identical, we conclude from (34) and (35) that for every vertex \( i_k \) there exists a \( j \) such that

\[
(g_{ijm})_p \geq 0 \quad \text{for all} \quad m = 0, 1, 2, \ldots, n_j. \tag{36}
\]

Further, the above result is valid irrespective of the port configuration.

It follows from (36) that in the case of \((n+2)\)-node \( n \)-port networks in which \( p = 2 \), all the conductances connecting vertices in \( T_1 \) to vertices in \( T_2 \) are non-negative. This result has already been established in [6].

Let \( N_d \) be the network of departure with respect to a given \( Y \) matrix and a port configuration \( T \). Let \( x_{(k),j} \) be the total number of positive conductances in \( N_d \) connecting vertex \( i_k \) to all the vertices in \( T_j \). Let

\[
x_{(k)} = \min \{ x_{(k),j}, j = 1, 2, \ldots, p, j \neq i \}. \tag{37}
\]

Theorem (3) then follows from (36).

**Theorem 3**

The number of conductances required for the realisation of a real symmetric matrix \( Y \) as the short-circuit conductance matrix of a resistive \( n \)-port network having no negative conductances and having a prescribed port configuration \( T \) cannot be less than

\[
1/2 \sum_{i=1}^{p} \sum_{k=0}^{n_i} x_{(k)}. \tag{38}
\]

V. Conclusion

The only approach available for the synthesis of resistive \( n \)-port networks having more than \((n+1)\)-nodes is due to Gillemin [8]. Gillemin's approach essentially requires the determination of a suitable padding \( n \)-port network \( N_p \) for a given network of departure \( N_d \) so that the parallel combination of \( N_d \) and \( N_p \) contains no negative conductances. Hence a number of procedures were suggested for generation of padding \( n \)-port networks [10, 11, 12]. All these procedures express conductances of a padding network in terms of certain arbitrary parameters. It was pointed out recently that a padding \( n \)-port network can be identified as the padding network.
of some resistive $n$-port network containing no negative conductances if and only if all $S_{ij}$'s are non-negative [6]. In view of this it is enough if we confine our search for a suitable padding $n$-port network to a restricted class of these networks. Further the potential factors and $S_{ij}$'s of a network and its padding network are identical. So, it seemed desirable and useful to develop a procedure for generating padding $n$-port networks in terms of these parameters. A step in this direction was taken in [6]. In Section III of this paper the approach presented in [6] is investigated and formulas for the conductances of a padding network in terms of potential factors and $S_{ij}$'s are obtained. Since these formulas are in terms of potential factors it is necessary that we know the necessary and sufficient conditions which the potential factors of a resistive $n$-port network having no negative conductances should satisfy. This necessity explains the interest in the analysis and the synthesis of the $K$-matrix of $n$-port networks.

In Section II, an equation relating the modified cutset matrix and the $K$-matrix is established. This relationship, as pointed out in Section II, is useful in view of the method used in [5] for determining the port structure pertinent to a given $K$-matrix.

The procedures for $K$-matrix synthesis given in [5] as well as in Section IV (a), of this paper require the solution of a linear program. However, the present approach involves a smaller number of unknowns and further the number of constraints used is also smaller for all $n > p (p - 2)$. Added to these is the usefulness in the $Y$-matrix synthesis problem.

Though the $Y$-matrix synthesis problem looks formidable, simultaneous realisation of $K$ and $Y$ matrices is straightforward as shown in Section IV (b). It is shown in [15] that the problem of synthesis of a hybrid matrix reduces to one of realising an $n$-port network having prescribed $K$ and $Y$ matrices and prescribed $S_{ij}$'s. Thus hybrid matrix synthesis can be achieved by a straightforward application of the results of this paper.

The lower bound on the number of conductances required for the realisation of a matrix $Y$ might help in throwing some light on Börzi's conjecture.

**Abstract**

In this paper a new matrix equation relating the modified cutset matrix $C$ and the potential factor matrix $K$, of an $n$-port network is first established. A new procedure for the synthesis of padding $n$-port networks is then given. Based on these results, a new necessary and sufficient condition is then established for the synthesis of the $K$-matrices of $n$-port networks. Application of these results in the synthesis of $Y$-matrices is discussed. A lower bound on the number of conductances required for $Y$-matrix synthesis is also given.

**Zusammenfassung**

In dem Artikel wird eine neue Matrizengleichung, die die modifizierte Schnittmengenmatrix $C$ und die Knotenspannungsmatrix $K$ eines $n$-Tor-Netzwerks miteinander verknüpft, vorgestellt. Ein neuer Algorithmus für die Synthese größerer $n$-Tor-Netzwerke wird gegeben. Mit diesen Ergebnissen wird eine neue hinreichende und notwendige Bedingung für die Synthese der $K$-Matrizen von $n$-Tor-Netzwerken abgeleitet. Die Ergebnisse werden bei der Synthese von $Y$-Matrizen angewendet und diskutiert.
References


