ON AN EXTREMAL PROBLEM IN GRAPH THEORY
AND ITS APPLICATION

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I. INTRODUCTION

The starting point for extremal problems in graph theory was the work of Turán \( \mathcal{T}_1,2 \). Some of the results available in this area of graph theory can be found in \( \mathcal{T}_3 \) and \( \mathcal{T}_4 \). Solution of extremal problems finds application in the design of optimally invulnerable communication nets. An approach usually followed in the vulnerability studies of communication nets is to define a meaningful vulnerability criterion and then relate the design of optimally invulnerable (with respect to the chosen criterion) communication nets, to the design of \( v \)-vertex \( e \)-edge graphs having a specified property. This approach has been followed in many of the results given \( \mathcal{T}_5 \). The results reported in \( \mathcal{T}_6 \) and in the present paper have been motivated by this consideration.

In \( \mathcal{T}_6 \), the concept of \( k \)-vulnerability of communication nets was introduced and the design of optimally \( k \)-invulnerable theory was the work of Turán \( \mathcal{T}_7 \). Some of the results related to the design of \( v \)-vertex \( e \)-edge graphs having the largest possible point covering number, and \( \mathcal{T}_8 \). Solution of extremal problems finds application.

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A new proof of Turan's result \( \lceil \frac{n}{2} \rceil \) on the minimum number of edges required to realize a \( v \)-vertex graph having a specified point covering number was given. An upper bound on the point covering number for a graph having \( v \)-vertices and \( e \)-edges was also established. In this paper, we consider the following dual problems:

1. Identification of maximum-edge \( v \)-vertex graphs having a specified edge independence number and having specified incidence relationship between matched and unmatched vertices.

2. Design of \( v \)-vertex \( e \)-edge graphs having the smallest edge independence number.

As we discuss the above problems we also establish a theorem due to Erdos and Gallai \( \lceil \frac{n}{2} \rceil \).

We now introduce the notation that will be followed in the paper.

\( G(V,E) \) will denote an undirected graph without parallel edges and self loops, where \( V \) is the set of vertices and \( E \) is the set of edges of the graph. \( (v_i, v_j) \) will denote the edge connecting vertices \( v_i \) and \( v_j \). Thus \( E \subseteq V \times V \).

The function \( f_g : V \times V \rightarrow \{0,1\} \) will be defined as follows:

\[
f_g(v_i, v_j) = 1, \text{ if } (v_i, v_j) \in E \text{ and } v_i \neq v_j \\
= 0, \text{ otherwise.}
\]
If $S$ and $T$ are mutually disjoint subsets of $V$, then

$$\{ (v_i, v_j) \mid v_i \in S, v_j \in T, v_i \neq v_j \}.$$  

Then we define

$$f_g(S,T) = \begin{cases} 1, & \text{if } (S,T) \subseteq E \\ 0, & \text{if } (S,T) \cap E = \emptyset \text{ the null set.} \end{cases}$$

For any set $X$, $|X|$ will denote the cardinality of $X$.

A set of edges in a graph is independent, if no two of them are adjacent.

The edge independent number $P_{\max}$ of a graph is the largest number of edges in any independent set of the graph.

An independent set of $P_{\max}$ edges of a graph is called a maximum matching of the graph.


II. MAXIMUM-EDGE v-VERTEX GRAPHS HAVING A SPECIFIED $P_{\max}$

Let $G(V,E)$ be a $v$-vertex graph with $P_{\max} = p$. Let

$$\{ e'_1, e'_2, \ldots, e'_p \}$$

be a set of $p$ independent edges in $G(V,E)$, where

$$e'_i = (a'_i, b'_i), i = 1, 2, \ldots, p.$$  

Let

$$A = \{ a'_1, a'_2, \ldots, a'_p \}$$

and

$$B = \{ b'_1, b'_2, \ldots, b'_p \}$$

then the set $(A \cup B)$ will represent the vertex set of the
edges \( e_1^i, e_2^i, \ldots, e_p^i \). Let \( V_b = V \setminus (A \cup B) \). We now define a partition \( \Pi = \{ V_0, V_1, V_2, V_0^*, V_1^*, V_2^* \} \) of the set \( (A \cup B) \) according to the following rules:

i) Let \( a_i \) be not adjacent to any vertex in \( V_b \). Then
   a) \( a_i \subseteq V_0 \) and \( b_i \subseteq V_0^* \) if \( b_i \) is not adjacent to any vertex in \( V_b \).
   b) \( a_i \subseteq V_1^* \) and \( b_i \subseteq V_1 \) if \( b_i \) is adjacent to exactly one vertex in \( V_b \).
   c) \( a_i \subseteq V_2^* \) and \( b_i \subseteq V_2 \) if \( b_i \) is adjacent to two or more vertices in \( V_b \).

ii) \( a_i \subseteq V_1 \), and \( b_i \subseteq V_1^* \) if \( a_i \) is adjacent to exactly one vertex in \( V_b \).

iii) \( a_i \subseteq V_2 \), \( b_i \subseteq V_2^* \) if \( a_i \) is adjacent to two or more vertices in \( V_b \).

Let, without loss of generality,

\[
V_2 = \{ v_1, v_2, \ldots, v_x \}, \quad |V_2| = x
\]

\[
V_1 = \{ v_{x+1}, v_{x+2}, \ldots, v_{x+y} \}, \quad |V_1| = y
\]

\[
V_0 = \{ v_{x+y+1}, v_{x+y+2}, \ldots, v_p \}, \quad |V_0| = p - (x+y)
\]

\[
V_2^* = \{ v_1^*, v_2^*, \ldots, v_x^* \}
\]

\[
V_1^* = \{ v_{x+1}^*, v_{x+2}^*, \ldots, v_{x+y}^* \}
\]

and

\[
V_0^* = \{ v_{x+y+1}^*, v_{x+y+2}^*, \ldots, v_p^* \}
\]

Let \( e_i = (v_i, v_i^*) \), \( i = 1, 2, \ldots, p \). It may be seen that the set of \( p \) edges \( e_1, e_2, \ldots, e_p \) is only a permutation of the set \( \{ e_1^i, e_2^i, \ldots, e_p^i \} \). Hence \( e_1, e_2, \ldots, e_p \) are also independent.
It may be observed that an I-pair with respect to \( v_i \) and \( v_j \) along with the set of \((p-2)\) independent edges
\[
\left\{ e_k \mid k = 1, 2, \ldots, p, \ k \neq i, \ k \neq j \right\}
\]
forms a set of \( p \) independent edges. This observation plays a significant role in the proofs of many of the lemmas that follow.

Through a series of lemmas, we now investigate the nature of the function \( f_g \). That is, we investigate whether a subset of \((V, X, V)\) is also a subset of \( E \) in \( G(V, E) \). Unless stated otherwise, all the discussions that follow are with respect to the graph \( G(V, E) \) defined at the beginning of this section. This graph will be referred to simply as \( G \).

**Lemma 1:**
\[
f_g(V_b, V_0^*) = f_g(V_b, V_0) = 0.
\]

**Proof:** This follows from the definition of \( V_0 \) and \( V_0^* \).

**Lemma 2:**

i) \( f_g(V_b, V_b) = 0 \)

ii) \( f_g(V_2^*, V_2^*) = 0 \)

iii) \( f_g(V_2^*, V_1^*) = 1 \)

iv) \( f_g(V_b, V_2^*) = 0 \)

**Proof:** We prove the lemma by contradiction.

i) Let, for some \( v_i, v_j \in V_b \)
\[
f_g(v_i, v_j) = 1.
\]
Then it may be seen that the \((p+1)\) edges
\[(v_i, v_j), e_1, e_2, \ldots, e_p\]
will form an independent set.
is contradicts the assumption that for \( G \), \( p_{\text{max}} = p \).

(3)

\[ f\left(v^*_i, v^*_j\right) = 1, \text{ for some } v^*_i, v^*_j \in V^*_2. \]

Then

(4)

\((p+1)\) edges \((v^*_i, v^*_j), e_1, e_2, \ldots, e_{i-1}, e_{i+1}, e_{i+2}, \ldots, e_p\) along with an \( I \)-pair of

and \( v_j \) will form an independent set of \((p+1)\) edges, contradicting the assumption that \( p_{\text{max}} = p \) for \( G \).

(6)

\[ f\left(v^*_i, v^*_j\right) = 1, \text{ for some } v^*_i \in V^*_2 \text{ and } v^*_j \in V^*_1. \]

A proof then proceeds in exactly the same way as in

(7)

i) above.

(8)

for some \( v^*_i \in V^*_b \) and \( v^*_j \in V^*_2 \), \( f\left(v^*_i, v^*_j\right) = 1, \)

on the \((p+1)\) edges \((v^*_i, v^*_j), e_1, e_2, \ldots, e_{j-1}, e_{j+1}, \ldots, e_p\) and \((v_j, v_1)\) where \( v_1 \in V^*_b (v_1 \neq v_i) \), will form

an independent set leading to a contradiction. //

\( (v-2p-1) \), all the edges in the set \( (V^*_b, V^*_1) \) will not

sent in \( G \). Let \( V^*_b, e_1, e_2, \ldots, e_{i+1}, e_{i+2} \).

By definition, each vertex in \( V^*_1 \) is connected to

"only one vertex in \( V^*_b \). Thus only \( \gamma \) of the \((v-2p)\)

in the set \( (V^*_b, V^*_1) \) will be present in \( G \), //

\[ n_c = \frac{v(v-1)}{2} \]

\[ n_c = \frac{(v-2p)(v-2p-1)}{2} = n^*_i (v, V^*_b, V^*_1) \]
Proof:

i) Let contrary to the lemma \( f_g(v_i, v_k^*) = 1 \), for some \( v_k^* \in (V_2^* \cup V_1^*), v_k \neq v_j^* \). Then it may be seen that the \((p-1)\) edges \((v_i^*, v_j^*), (v_i, v_k^*), e_1, e_2, \ldots, e_{j-1}, e_{j+1}, \ldots, e_{k-1}, e_{k+1}, \ldots, e_p\) along with an I-pair of \( v_j \) and \( v_k \) will form a set of \((p+1)\) independent edges. Hence a contradiction.

ii) From (i) above,

\[
f_g(v_i, v_k^*) = 0, \forall v_k^* \in V_2^* \cup V_1^*, v_k^* \neq v_j^* \quad \text{and} \quad f_g(v_i, v_k^*) = 0, \forall v_k^* \in V_2^* \cup V_1^*, v_k^* \neq v_1^*\]

Hence

\[
f_g(v_i, v_k^*) = 0, \forall v_k^* \in V_2^* \cup V_1^*\]

i.e.,

\[
f_g(v_i, V_2^* \cup V_1^*) = 0\]

Hence the proof. \(//\)

Theorem 2:

\[n_{x,y} \leq n_c - n_a - xz, \text{ if } x + y \geq 2, x \neq 0.\]

Proof:

Let \( z_i, i = 1, 2, \ldots, l \) be the number of vertices in \( V_1^* \) which are adjacent to \( i \) vertices in \( V_2^* \). Then it follows from theorem (1) and lemma (4) that

\[
n_{x,y} \leq n_c - n_a - \sum_{i=1}^{l} z_i(x-i) - z_1(x+y-1) - \sum_{i=2}^{l} z_i(x+y) - (z-z_1-z_2, \ldots, z_l)x
\]

\[= n_c - n_a - z(x-z_1(x+y-2) - \sum_{i=2}^{l} z_i(x+y-i))\]

\(\therefore n_c - n_a - zx \quad //\)
Lemma 5:
\[ f_g(v_i, v_j) = 0, \ v_i \in V_b, \ v_j \in V_1 \]
\[ \implies f_g(v_i, v_j^*) = 0. \]

Proof:
If contrary to the lemma \( f_g(v_i, v_j^*) = 1 \), then it may be seen that the \((p+1)\) edges \( e_1, e_2, \ldots, e_{j-1}, e_{j+1}, \ldots, e_p \), \((v_i, v_j), (v_j, v_1)\) where \( v_1 \in V_b \), will form an independent set contradicting the assumption that \( p_{\text{max}} = p \) for \( G \). //

Lemma 6:
\[ f_g(v_i, v_j^*) = 1, \ v_i \in V_b, \ v_j^* \in V_1^* \]
\[ \implies f_g(v_j, v_i) = 1. \]

Proof:
If \( f_g(v_j, v_i) = 0 \), then there exists a \( v_k \in V_b \) such that \( f_g(v_j, v_k) = 1 \). Hence the \((p+1)\) edges \((v_i, v_j^*), (v_j, v_k), e_1, e_2, \ldots, e_{j-1}, e_{j+1}, \ldots, e_p\) will form an independent set contradicting the fact that \( p_{\text{max}} = p \) for \( G \). //

Lemma 7:
\[ f_g(v_i, v_j^*) = 1, \ v_i \in V_b, \ v_j^* \in V_1^* \]
\[ \implies f_g(v_j^*, v_k) = 0, \ \forall v_k \in V_b, \ v_k \neq v_i. \]

Proof:
It follows from the previous lemma that
\[ f_g(v_j, v_i) = 1, \ v_i \in V_b, \ v_j \in V_1 \] if \( f_g(v_i, v_j^*) = 1 \). Hence
\[ f_g(v_j, v_k) = 0, \ \forall v_k \in V_b, \ v_k \neq v_i. \] This result, together with lemma 5, implies
\[ f_g(v_j^*, v_k) = 0, \ \forall v_k \in V_b, \ v_k \neq v_i. \]
Let \( f_g(v_i, v_j) = 1 \), \( v_i \in V_b \), \( v_j \in V_1^* \).

\[ f_g(v_j, V_2^*) = 0. \]

If \( f_g(v_i, v_j^*) = 1 \), \( v_i \in V_b \) and \( v_j^* \in V_1^* \), then by a 6

\[ f_g(v_i, v_j) = 1 \]

contrary to the lemma \( f_g(v_j, v_k^*) = 1 \), for some \( v_k^* \in V_2^* \).

Further, \( f_g(v_k, v_1) = 1 \), \( v_k \in V_b \), \( v_k \neq v_i \). Then the edges \((v_i, v_j^*), (v_j, v_k^*), (v_k, v_1), e_1, \ldots, e_{k-1}, e_{k+1}, e_{j-1}, e_{j+1}, \ldots, e_p\) will form an independent set leading to a contradiction.

*rem_3:*

If \( x + y \geq 2 \), and \( x \neq 0 \), then

\[ n_{x, y} \leq \frac{p(p-1)}{2} + x(v-2p) + p^2 + \frac{(p-x)(p-x-1)}{2} + y. \]

*if:*

It follows from lemma 7 that only \( r \leq y \) of the \(-2p\) edges in the set \((V_b, V_1^*)\) will be present in \( G \).

Further, if \( f_g(v_i, v_j^*) = 1 \), \( v_i \in V_b \), and \( v_j^* \in V_1^* \), by lemma 8, \( f_g(v_j, V_2^*) = 0 \).

Hence by using theorem 2 and the above facts, we get

\[ n \leq n_c - n_a - xz - rx - \left\{ \frac{1}{2} y(v-2p) - r \right\} \]

\[ = n_c - n_a - y(v-2p) - r(x-1) - zx. \]
Proof:

According to lemma (1),

\[ f(V_b, V_b) = f(V_b, V_0) = f(V_b, V_0) = 0. \]

If \( x = 0 \), then \( V_2 = \emptyset \), the null-set. From the definition of \( V_1 \), we see that only \( y(v-2p) \) edges in the set \( (V_b, V_1) \) will be present in \( G \). Further it follows from lemma 7 that there are at most \( y \) edges which connect vertices in \( V_1^* \) to those in \( V_b \). These results lead to the following:

Thus

\[ n_{0,y} \leq \frac{2p(2p-1)}{2} + 2y. \]

Theorem 6:

\[ n_{0,y} \leq n_{0,p} = \frac{2p(2p-1)}{2} + 2p \]

Proof:

It follows from theorem 5 that

\[ f(n_0, y) \leq f(n_0, p) = \frac{2p(2p-1)}{2} + 2p = 0. \]

If \( y = 0 \), then \( V_2 = \emptyset \), the null-set. From the definition of \( V_1 \), the graph shown in Fig. 2 has \( p_{max} = p \), \( x = 0 \), \( y = p \) and \( \emptyset \). Since it has \( \frac{2p(2p-1)}{2} + 2p \) edges it follows that

\[ n_{0,p} \leq n_{0,\bar{p}} = \frac{2p(2p-1)}{2} + 2p. \]

Thus

and the graph of Fig. 2 is a \( G_{n, p} \) graph. It may be seen that this graph consists of a complete subgraph on \( (2p+1) \) vertices with all the other vertices isolated.
Theorem 9:

i) \( n_{p,0} \geq \propto \) for \( v \geq \frac{5p + 3}{2} \)

ii) \( n_{0,p} \geq \propto \) for \( v \leq \frac{5p + 3}{2} \)

iii) \( n_{p,0} \geq n_{1,0} \) iff \( v \geq \frac{5p}{2} \)

iv) \( n_{p,0} \geq n_{0,p} \) iff \( v \geq \frac{5p + 3}{2} \)

v) \( n_{1,0} \geq n_{0,p} \) iff \( v \geq 4p \).

vi) \( n_{0,p} \geq n_{0,0} \)

Proof:
Proofs of (i) and (ii) are given in the Appendix. The other results follow after some straightforward manipulations. //

The main results of the paper follow next.

Theorem 10: \( \Omega \)

a) If \( |p| \leq \frac{5p + 3}{2} \), then \( G_{0,p} \) is a max-edge \( v \)-vertex graph with \( p_{\text{max}} = p^* \).

b) If \( v \geq \frac{5p + 3}{2} \), then \( G_{p,0} \) is a max-edge \( v \)-vertex graph with \( p_{\text{max}} = p^* \).

Proof:
It may be easily verified from theorem 9, that for \( v \leq \frac{5p + 3}{2} \), \( n_{0,p} \) is greater than \( \propto \), \( n_{1,0} \), \( n_{p,0} \) and \( n_{0,0} \).
Similarly for $v \geq \frac{5p+3}{2}$, $n_{p,0}$ is greater than $\alpha$, $n_{1,0}$, $n_{0,p}$ and $n_{0,0}$. Hence the result. //

Of the two optimal graphs $G_{p,0}$ and $G_{0,p}$, the latter is connected. In the vulnerability studies of communication nets only connected graphs will be of interest. Hence, in such a case, we may ignore $G_{0,p}$ and look for connected optimal graphs. It is shown in the Appendix that

$$n_{1,0} \geq \alpha \quad \text{if} \quad v \leq \frac{5p}{2}$$

and

$$n_{p,0} \quad \text{if} \quad v \geq \frac{5p}{2}$$

It follows from theorem 9, that $n_{1,0} \geq n_{p,0}$ if $v \leq \frac{5p}{2}$.

Using these inequalities, we obtain the following main theorem applicable for connected graphs.

**Theorem 11:**

a) If $v \leq \frac{5p}{2}$, $G_{1,0}$ is a max-edge $v$-vertex connected graph with $p_{\max} = p$.

b) If $v \geq \frac{5p}{2}$, $G_{p,0}$ is a max-edge $v$-vertex connected graph with $p_{\max} = p$.

**III. DESIGN OF $v$-VERTEX $e$-EDGE GRAPHS HAVING THE SMALLEST $p_{\max}$**

In this section we first obtain a lower bound on $p_{\max}$ given the number of vertices and the number of edges. This bound will then be used to design $v$-vertex $e$-edge graphs having the smallest $p_{\max}$. Let the functions $\rho_i$, $i = 1, 2, \ldots, 5$ be defined as
It follows from theorem 10(b) that

\[ \rho_i(p_i, v) \geq \rho_i(p_i, v), \quad \therefore \quad v \geq \frac{5p_i + 3}{2} \]

Since \( \rho_i(p_i, v) \geq e \), this means

\[ \rho_i(p_i, v) \geq e \]

The above inequality contradicts the definition that \( p_1 \) is the smallest value of \( p \) satisfying

\[ \rho_1(p, v) \geq e \]

Hence \( p_i \geq p_1 \) for all \( i = 1, 2, \ldots, 5 \).

b) The proof of part (b) of the theorem follows in a similar manner from theorem 10(a).

Theorem 13:

a) If \( p_1 \leq \frac{2v}{5} \), then

\[ p_1 = \text{Min} \left\{ p_3, p_4, p_5 \right\} \]

b) If \( p_3 \geq \frac{2v}{5} \), then

\[ p_3 = \text{Min} \left\{ p_1, p_4, p_5 \right\} \]

Proof: The proof follows from theorem 11.

It is clear from theorem 13 that

\[ \rho_i(p_i, v) \geq e \quad (12) \]

or

\[ \rho_i(p_i, v) \geq e \quad (13) \]
Remove from $G_{p,0}$ any $(n_{p,0} - e)$ edges making sure that the resulting graph will be connected and has $p_{\text{max}} = p$.

This can be achieved, for example, by retaining the edges $(v_1^*, v_1), (v_1, v_2^*), (v_2^*, v_2), \ldots, (v_p^*, v_p)$ and the set $(v_1, V_b)$.

**Case-2:** If $e \geq \frac{(4v-1)(2v-3)}{25}$

and $e \leq \frac{8v^2 - 5v}{25}$

and a connected graph is required, repeat case-1.

**Case-3:** If $e \geq \frac{(4v-1)(2v-3)}{25}$ and a connected graph is not required then calculate $p_2$, using (15). Let $p = [p_2]^*$. Construct $G_{0,p}$. Remove from $G_{0,p}$ any $(n_{0,p} - e)$ edges making sure that the resulting graph has $p_{\text{max}} = p$.

**Case-4:** If $e \geq \frac{8v^2 - 5v}{25}$ and a connected graph is required, calculate $p_3$ using (16) and let $p = [p_3]^*$. Construct $G_{1,0}$ having $p_{\text{max}} = p$ and remove from $G_{1,0}$ any $(n_{1,0} - e)$ edges making sure that the resulting graph has $p_{\text{max}} = p$ and remains connected.

**IV. CONCLUSIONS**

In this paper we have considered the following problems:

1. Identification of max-edge $v$-vertex graphs having a specified edge independence number (Erdős and Gallai theorem $\leftarrow 8 \rightarrow$).
Substituting $v = \frac{5p + 3}{2}$ in the above we get

$$n_{0, p} - \kappa \geq \left( -\frac{p}{2} - \frac{x^2}{2} \right) + 2p - (2x + y), \text{ for } v \leq \frac{5p+3}{2}$$

\[\geq 0 \text{ since } p \geq x+y\]

Hence the proof.

iv) $n_{1, 0} - \kappa = (v-p) + \frac{2p(2p-1)}{2} - p^2 - \frac{p(p-1)}{2} - \frac{(p-x)(p-x-1)}{2}$

\[-y - x(v-2p)\]

\[= (x-1)(-v + 2p + \frac{x}{2} + p-x)\]

Substituting $v = \frac{5p}{2}$ in the above we get

$$n_{1, 0} - \kappa \geq (x-1)(-\frac{p}{2} + \frac{x}{2} + p - x), \text{ for } v \leq \frac{5p}{2}$$

\[\geq 0 \text{ since } x \geq \frac{p}{2}\]

Hence the proof.