ON THE NUMBER OF CONDUCTANCES REQUIRED FOR REALIZING Y AND K MATRICES

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SUMMARY

Upper bounds are established on the number of conductances required for realizing a real symmetric matrix $Y$ as the short-circuit conductance matrix of a resistive $n$-port network containing no negative conductances, and for the realization of a real matrix $K$ as the potential factor matrix of a similar network without negative conductances. These results are the consequence of the properties of the modified cut-set matrix of an $n$-port and a theorem in the theory of linear programming.

1. INTRODUCTION

Biorci$^{1,2}$ conjectured that, at most $n(n+1)/2$ conductances are required for realizing a real symmetric matrix as the short-circuit conductance matrix of a resistive $n$-port network containing no negative conductances. Even after several years of research, this conjecture has been neither proved nor disproved. However, a lower bound is known for the realization of $Y$ matrices when the port configuration of the required network is specified.$^3$ In this paper, we establish upper bounds on the number of conductances required for realizing $Y$ and $K$ matrices. These results are the consequence of the properties of the modified cut-set matrix of an $n$-port and a theorem in the theory of linear programming.

2. AN UPPER BOUND ON THE NUMBER OF CONDUCTANCES REQUIRED FOR REALIZING A $Y$ MATRIX

In this Section, we first summarize some results relating to the modified cut-set matrix of a resistive $n$-port network$^4$ and also state a theorem in the theory of linear programming. These results are then used to establish an upper bound on the number of conductances required for realizing an $(n \times n)$ $Y$ matrix by an $(n + p)$-node $n$-port network.

Consider a resistive $n$-port network $N$. Let the port configuration $T$ of $N$ be in $p$ connected parts $T_1, T_2, \ldots, T_p$. Permitting edges with zero conductances, the graph of $N$ can be considered to be complete. Let $T_0$ be a tree of $N$ such that $T \subseteq T_0$. The branches of $T$ will be called the port branches, and the remaining branches of $T_0$ will be referred to as the non-port branches.

Let $C_0$, the fundamental cut-set matrix of $N$ with respect to the tree $T_0$, be partitioned as follows:

$$C_0 = \begin{bmatrix} C_{1:} \\ C_{2:} \end{bmatrix}$$

(1)

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where the rows of $C_1$ correspond to the port branches and those of $C_2$ correspond to the non-port branches.

The cut-set admittance matrix $Y_0$ of $N$ with respect to the tree $T_0$ is defined as

$$Y_0 = C_0 G C_0^t$$

$$= \begin{bmatrix} C_1 & G & C_1 \mid C_1 & G & C_2 \end{bmatrix} \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix}$$

(2)

where $G$ is the diagonal matrix of edge conductances of $N$. The short-circuit conductance matrix $Y$ of $N$ is given by

$$Y = Y_{11} - Y_{12} Y_{22}^{-1} Y_{21}$$

(3)

The modified cut-set matrix of $N$ is defined as

$$C = C_1 - Y_{12} Y_{22}^{-1} C_2$$

(4)

The following results have been proved in Reference 4:

**Theorem 1**

Let $C$ be the modified cut-set matrix of a connected resistive $n$-port network $N$ having a port configuration $T$. Let $C_0$ be the fundamental cut-set matrix of $N$ with respect to a tree $T_0$ of which $T$ is a subgraph. Further let $C_1$ and $C_2$, the submatrices of $C_0$, correspond respectively to the port branches and the non-port branches of $T_0$. Let $Y$ be the short-circuit conductance matrix of $N$ with respect to the port configuration $T$.

(a) If $G'$ is the diagonal matrix of edge conductances of a connected $n$-port network $N^*$ having the same port configuration as that of $N$ and $CG' C_2 = 0$, then the modified cut-set matrix of $N^*$ is also equal to $C$.

(b) Let

$$CG' C_1 = y$$

and

$$CG' C_2 = 0$$

where $G'$ is the diagonal matrix of edge conductances of an $n$-port network $N^*$ having the same port configuration as that of $N$. Then the modified cut-set matrix and the short-circuit conductance matrix of $N^*$ are equal to $C$ and $Y$, respectively.

**Theorem 2**

Two $n$-port networks have the same modified cut-set matrix if they have the same $K$ matrix.

Consider next the following set of $m$ simultaneous equations in $n$ variables $x_1, x_2, \ldots, x_n$:

$$AX = b$$

(5)

where $A$ is an $(m \times n)$ real matrix, $X$ is the column vector of the variables $x_1, x_2, \ldots, x_n$ and $b$ is a column vector of real elements.

Any nonnegative solution of (5) is called a feasible solution. If any $(m \times m)$ nonsingular matrix is chosen from $A$, and if all the $(n-m)$ columns of this matrix are set equal to zero, the solution to the resulting system of equations is called a basic solution. If a basic solution is feasible, then it is called a basic feasible solution. Thus the number of nonzero variables in a basic feasible solution will be less than or equal to $m$, the number of equations. The following result is proved in Reference 6.

**Theorem 3**

Consider a set of $m$ simultaneous equations in $n$ variables ($n \geq m$)

$$Ax = b$$
If there exists a feasible solution \( x \geq 0 \) to these equations, then there exists a basic feasible solution.
We now prove the following theorem:

**Theorem 4**

If a matrix \( Y \) is realizable as the short-circuit conductance matrix of an \((n + p)\)-node resistive \( n \)-port, then it can be realized by an \( n \)-port network containing at most \( m = \{ n(n + 1)/2 + n(p - 1) \} \) conductances.

**Proof**

Let the matrix \( Y \) be the short-circuit conductance matrix of an \((n + p)\)-node \( n \)-port network contains \( m \) or less number of conductances, the theorem is proved. Otherwise, we proceed as follows to obtain an equivalent network containing, at most, \( m \) conductances.

Let \( C \) be the modified cut-set matrix of \( N_1 \). Let \( C_1 \) and \( C_2 \) be defined as in Theorem 1. Let \( G_1 \) be the diagonal matrix of edge conductances of \( N_1 \).

Consider the following sets of equations:

\[
\begin{align*}
CGC_1 &= 0 \quad (6a) \\
CGC_1 &= Y \quad (6b)
\end{align*}
\]

Note that each one of the matrices \( C \) and \( C_1 \) has \( n \) rows and the matrix \( C_2 \) has \( (p - 1) \) rows. Also the number of variables in \( G \) is equal to \( l \) where \( l = (n + p)(n + p - 1)/2 \).

Hence, equation (6a) represents a set of \( n(p - 1) \) equations in \( l \) variables. Further, because of the symmetry of \( Y \), equation (6b) represents a set of \( n(n + 1)/2 \) equations in \( l \) variables. Thus equations (6) represent a set of \( m \) equations in \( l \) variables.

The edge-conductance matrix \( G_1 \) of the network \( N_1 \) is a feasible solution of (6). Hence, there exists a basic feasible solution \( G \). The number of nonzero variables in \( G_2 \) is less than or equal to \( m \). Since, by Theorem 1(b), \( G_2 \) is the matrix of conductances of an \( n \)-port network \( N_2 \) whose short-circuit conductance matrix is equal to \( Y \), we conclude that, for the given matrix \( Y \), there exists an \((n + p)\)-node realization containing, at most, \( m \) conductances.

**Example 1**

The matrix \( Y \) given below is the short-circuit conductance matrix of a 3-port network \( N_1 \) having the port configuration \( T \) shown in Figure 1.

\[
Y = \begin{bmatrix}
1.00 & -0.08 & -0.08 \\
-0.08 & 2.00 & 0.08 \\
-0.08 & 0.08 & 3.00
\end{bmatrix}
\]

![Figure 1. Port configuration for Example 1](image)

The diagonal matrix \( G_1 \) of edge conductances (all in siemens) of \( N_1 \) is given by

\[
G_1 = \text{diag} \{ g_{12}, g_{13}, g_{14}, g_{15}, g_{16}, g_{23}, g_{24}, g_{25}, g_{26}, g_{34}, g_{35}, g_{36}, g_{45}, g_{46}, g_{56} \}
\]

\[
= \text{diag} \{ 0.49, 0.06, 0.14, 0.45, 0.05, 0.54, 1.26, 0.45, 0.05, 1.08, 0.70, 0.70, 0.30, 0.30, 2.33 \}
\]
The modified cut-set matrix $C$ of $N_1$ is obtained as follows:

$$C = \begin{bmatrix}
  1 & 0.8 & 0.8 & 0.7 & 0.7 & -0.2 & -0.2 & -0.3 & -0.3 & 0 & -0.1 & -0.1 & -0.1 & 0 \\
  0 & -0.6 & 0.4 & -0.2 & -0.2 & -0.6 & 0.4 & -0.2 & 1 & 0.4 & 0.4 & -0.6 & -0.6 & 0 \\
  0 & 0.1 & 0.1 & -0.3 & 0.7 & 0.1 & 0.1 & -0.3 & 0.7 & 0 & -0.4 & 0.6 & -0.4 & 0.6 & 1
\end{bmatrix}$$

Choosing the edges $e_{23}$ and $e_{45}$ as the nonport branches, we obtain the matrices $C_1$ and $C_2$ as follows:

$$C_1 = \begin{bmatrix}
  1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1
\end{bmatrix}$$

$$C_2 = \begin{bmatrix}
  0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0
\end{bmatrix}$$

A basic feasible solution $G_2$ for the set of equations

$$CGC_1^T = Y$$

and

$$CGC_2^T = 0$$

is then obtained using the MPS package available with the IBM 370/155 computer system.

The nonzero entries of $G_2$ are as follows:

$$g_{12} = 0.64800, \ g_{15} = 0.32000, \ g_{25} = 0.42286, \ g_{35} = 0.17143$$
$$g_{13} = 0.08000, \ g_{23} = 0.14857, \ g_{26} = 0.17143, \ g_{36} = 0.28571$$
$$g_{14} = 0.08000, \ g_{24} = 0.72000, \ g_{34} = 1.68000, \ g_{56} = 2.70857$$

For the case under consideration, $n = 3$ and $p = 3$, and so $m = 12$. Note that the number of nonzero entries in $G_2$ is equal to 12. Thus the 3-port network $N_2$ of which $G_2$ is the matrix of edge conductances is a realization of the given matrix $Y$ containing, at most, $m$ conductances.

3. AN UPPER BOUND ON THE NUMBER OF CONDUCTANCES REQUIRED FOR REALIZING A $K$ MATRIX

In this Section, we establish an upper bound on the number of conductances required for the realization of a real matrix $K$ as the potential factor matrix of an $(n+p)$-node $n$-port resistive network containing no negative conductances.

**Theorem 5**

If a real matrix $K$ is realizable as the potential factor matrix of an $(n+p)$-node $n$-port network then it can be realized by an $n$-port network containing, at most, $(n(p-1)+(p-1))$ conductances.

**Proof**

Let the given matrix $K$ be the potential factor matrix of an $(n+p)$-node $n$-port network $N_1$. If $N_1$ contains $(n(p-1)+(p-1))$ or less conductances, the theorem is proved. Otherwise, we proceed as follows to obtain an equivalent $n$-port network $N_2$ containing, at most, $(n(p-1)+(p-1))$ conductances.
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Let \( C \) be the modified cut-set matrix of the \( n \)-port network \( N_1 \) realizing the given \( K \) matrix. Let \( G_1 \) be the diagonal matrix of edge conductances of \( N_1 \). Let the matrix \( G_2 \) be defined as in Theorem 1.

Consider any diagonal matrix \( G_2 \) of real nonnegative entries satisfying the equation

\[
CG_2G_1^T = 0
\]  

(7)

Let \( G_2 \) be the matrix of edge conductances of a connected \( (n + p) \)-node \( n \)-port network \( N_2 \). Then, by Theorem 1a, the modified cut-set matrix of \( N_2 \) is equal to \( C \). Also, by Theorem 2, the potential factor matrix of \( N_2 \) is equal to the matrix \( K \). To ensure that a solution \( G_2 \) of (7) corresponds to a connected \( n \)-port network, we proceed as follows:

Let the \( p \) connected parts of the port configuration of \( N_1 \) be denoted by \( T_1, T_2, \ldots, T_p \). Let \( (S_q)_1 \) denote the sum of the conductances in the given network \( N_1 \) connecting vertices in \( T_i \) to those in \( T_j \). \( (S_q)_2 \) will refer to the corresponding quantity in the required network \( N_2 \). Note that the port configuration of \( N_2 \) will be the same as that of \( N_1 \).

If all the ports of \( N_2 \) are short-circuited, the network \( (N_2)_S \) that results will have \( p \) vertices. \( (S_q)_S \) will represent the different conductances of \( (N_2)_S \). If \( (N_2)_S \) is connected, \( N_2 \) will also be connected.

Choose a set of \( (p - 1) \) positive conductances \( (S_q)_1 \)'s such that they constitute a tree of \( (N_1)_S \). Let these conductances be denoted by

\[
(S_{q_{1k_1}}), (S_{q_{2k_2}}), \ldots, (S_{q_{p-1k_{p-1}}})
\]

If the corresponding conductances of \( (N_2)_S \) are also positive, then, as mentioned earlier, the \( n \)-port network \( N_2 \) will be connected.

Consider then the following set of \( (p - 1) \) equations:

\[
(S_{q_{jk}}) = (S_{q_{jk}})_1 \quad j = 1, 2, \ldots, p - 1
\]  

(8)

Note that each \( (S_{q_{jk}})_1 \) can be written as a sum of the entries of the matrix \( G \).

Any solution of (7) and (8) will correspond to the diagonal matrix of edge conductances of a connected \( n \)-port network.

Equations (7) and (8) together represent a set of \( np(p - 1) + (p - 1) \) equations in \( (n + p)(n + p - 1)/2 \) variables. \( G_1 \), the diagonal matrix of edge conductances of \( N_1 \), is a feasible solution of these equations. Hence a basic feasible solution \( G_2 \) exists. The number of nonzero conductances in this basic feasible solution is less than or equal to \( np(p - 1) + (p - 1) \). Thus there exists a network \( N_2 \) (of which \( G_2 \) is the diagonal matrix of edge conductances) containing, at most, \( np(p - 1) + (p - 1) \) conductances. As stated earlier, the network \( N_2 \) will realize the given matrix \( K \). Hence the theorem.

Example 2

The matrix \( K \) given below is the potential factor matrix of a 4-port network \( N_1 \) having the port configuration shown in Figure 2.

\[
K = \begin{bmatrix}
1 & 1 & 1 & \frac{7}{9} \\
0 & 1 & \frac{5}{9} & \frac{1}{9} \\
0 & 0 & 1 & \frac{3}{9} \\
\frac{5}{9} & \frac{5}{9} & \frac{5}{9} & 1
\end{bmatrix}
\]

![Figure 2. Port configuration for Example 2](image-url)
The matrix \( G_1 \) of edge conductances (all in siemens) of \( N_1 \) is given by
\[
G_1 = \text{diag} \begin{bmatrix} 8_{12} & 8_{13} & 8_{14} & 8_{15} & 8_{16} & 8_{23} & 8_{24} & 8_{25} \\
8_{26} & 8_{34} & 8_{35} & 8_{36} & 8_{45} & 8_{46} & 8_{56} \end{bmatrix}
= \text{diag} \begin{bmatrix} \frac{9}{5} & \frac{1}{2} & \frac{2}{9} & \frac{1}{6} & \frac{10}{9} & \frac{6}{9} & \frac{3}{5} & \frac{9}{9} \\
\frac{9}{9} & \frac{7}{9} & \frac{3}{9} & \frac{1}{9} & \frac{16}{9} & \frac{12}{9} & \frac{4}{9} & \frac{8}{9} \end{bmatrix}
\]

The modified cut-set matrix \( C \) of \( N_1 \) is obtained as follows:
\[
C = \begin{bmatrix}
1 & 1 & 1 & \frac{8}{9} & \frac{3}{9} & 0 & 0 & -\frac{9}{9} & -\frac{9}{9} & 0 & -\frac{9}{9} & -\frac{9}{9} & -\frac{9}{9} & 0 \\
0 & 0 & 1 & \frac{9}{6} & \frac{3}{9} & 1 & 1 & \frac{9}{9} & 0 & -\frac{9}{9} & -\frac{9}{9} & -\frac{9}{9} & 0 \\
0 & 0 & 0 & -\frac{9}{9} & \frac{9}{9} & 0 & 0 & -\frac{9}{9} & -\frac{9}{9} & 0 & -\frac{9}{9} & -\frac{9}{9} & 1
\end{bmatrix}
\]

Choosing the edge \( e_{45} \) connecting the vertices 4 and 5 as the nonport branch we obtain \( G_2 \) as follows:
\[
G_2 = \begin{bmatrix} 8_{12} & 8_{13} & 8_{14} & 8_{15} & 8_{16} & 8_{23} & 8_{24} & 8_{25} & 8_{26} & 8_{34} & 8_{35} & 8_{36} & 8_{45} & 8_{46} & 8_{56} & 0 \end{bmatrix}
= \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}
\]

In \( (N_1)_S, S_{12} \), the combination of the conductances \( g_{15}, g_{16}, g_{25}, g_{26}, g_{35}, g_{36}, g_{45} \) and \( g_{46} \) forms a tree. A basic feasible solution \( G_2 \) to the following sets of equations is required.
\[
CG_2G_2^T = 0
\]
\[
(S_{12}) = (S_{12})_1 \quad \text{i.e.,} = 9
\]

After substituting for \( C \) and \( C_2 \), the above simplifies to the following:
\[
\begin{bmatrix}
0 & 0 & 0 & 7 & 7 & 0 & 0 & -2 & -2 & 0 & -2 & -2 & -2 & -2 & 0 \\
0 & 0 & 0 & 5 & 5 & 0 & 0 & 5 & 5 & 0 & -4 & -4 & -4 & -4 & 0 \\
0 & 0 & 0 & 3 & 3 & 0 & 0 & 3 & 3 & 0 & 3 & 3 & -6 & -6 & 0 \\
0 & 0 & 0 & -5 & 4 & 0 & 0 & -5 & 4 & 0 & -5 & 4 & -5 & 4 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0
\end{bmatrix}
= \begin{bmatrix} 0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
9 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \end{bmatrix}
\]

Using the MPS package, the following basic feasible solution \( G_2 \) is obtained. The nonzero entries of \( G_2 \) (all in siemens) are given by
\[
g_{16} = 2.0 \quad g_{25} = 2.0 \quad g_{36} = 2.0 \quad g_{45} = 2.0 \quad g_{46} = 1.0
\]
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Note that, in this case, \( n = 4 \) and \( p = 2 \). Hence \( n(p-1)+(p-1) = 5 \). It may be seen that \( G_2 \) contains five nonzero entries. The network \( N_2 \) of which \( G_2 \) is the diagonal matrix of edge conductances is a realization of the matrix \( K \) containing \( n(p-1)+(p-1) \) conductances.

4. CONCLUSIONS

In this paper, we have established upper bounds on the number of conductances required for realizing \( Y \) and \( K \) matrices. According to Theorem 4, the maximum number of conductances required for realizing any \((n \times n)\) \( Y \) matrix by an \((n+2)\)-node \( n \)-port network is equal to \( n(n+1)/2+n \). In a recent paper, it was shown that any \( Y \) matrix realizable by an \((n+1)\)-node \( n \)-port network containing no zero conductances can be realized by an \( n \)-port network containing, at most, \( n(n+1)/2+1 \) conductances, which is less than the maximum number of conductances required according to Theorem 5. It may, therefore, be expected that the approach of Reference 7 can be generalized to obtain \((n+p)\)-node realizations of \( Y \) matrices of \((n+1)\)-node \( n \)-port networks containing, at most, \( n(n+1)/2(p-1)/2 \) conductances.

REFERENCES