Applied Logic for Hardware and Software

Software Properties as Axioms and Theorems
proofs by induction
... or ...
How to Write Reliable Software
and know it's reliable
Example: Multiplexor Function

Problem: Multiplex two sequences to form a new sequence
- \( \text{mux } [1, 2, 3, 4, 5] [6, 7, 8, 9, 10] = [1, 6, 2, 7, 3, 8, 4, 9, 5, 10] \)
- \( \text{mux } [1, 2, 3, 4, 5] [6, 7, 8] = [1, 6, 2, 7, 3, 8, 4, 5] \)
- \( \text{mux } [1, 2, 3, 4] [6, 7, 8, 9, 10] = [1, 6, 2, 7, 3, 8, 4, 9, 10] \)

Notation
- \([x_1, x_2, x_3]\) \text{ denotes a sequence with three elements}
- \(xs ++ ys\) \text{ denotes concatenation of the sequences } xs \text{ and } ys
  \(\sqrt{[x_1, x_2, x_3] ++ [y_1, y_2, y_3, y_4]} = [x_1, x_2, x_3, y_1, y_2, y_3, y_4]\)

Informal specification of \text{mux}
- \(\text{mux } [x_1, x_2, \ldots x_n] [y_1, y_2, \ldots y_n] = [x_1, y_1, x_2, y_2, \ldots x_n, y_n]\)
- Note: append extra elements in either sequence to the end
Multiplexor Function — Testing

Problem: multiplex two sequences to form a new sequence
- \text{mux} \ [1, 2, 3, 4, 5] \ [6, 7, 8, 9, 10] = \ [1, 6, 2, 7, 3, 8, 4, 9, 5, 10]
- \text{mux} \ [1, 2, 3, 4, 5] \ [6, 7, 8] = \ [1, 6, 2, 7, 3, 8, 4, 5]
- \text{mux} \ [1, 2, 3, 4] \ [6, 7, 8, 9, 10] = \ [1, 6, 2, 7, 3, 8, 4, 9, 10]

Notation
- \ [x_1, x_2, x_3] \leftarrow \text{denotes a sequence with three elements}
- \text{xs ++ ys} \leftarrow \text{denotes concatenation of the sequences xs and ys}
  \[ [x_1, x_2, x_3] ++ [y_1, y_2, y_3, y_4] = [x_1, x_2, x_3, y_1, y_2, y_3, y_4] \]

Informal specification of \text{mux}
- \text{mux} \ [x_1, x_2, \ldots x_n] \ [y_1, y_2, \ldots y_n] = [x_1, y_1, x_2, y_2, \ldots x_n, y_n]
  \text{Note: append extra elements in either sequence to the end}

Suppose we've defined \text{mux} in a program and want to test it
- Example test: \text{mux} \ [1, 2, 3] \ [4, 5, 6, 7] == \ [1, 4, 2, 5, 3, 6, 7]

That's an okay test, but hard to generalize for automated testing
- The following tests are \text{wrong} with lists of differing lengths
  \[ \text{muxTest1 xs ys} = ((\text{mux xs ys})_{2n+1} == \text{xs}_n \text{ if } n \leq \text{length xs}) \]
  \[ \text{muxTest2 xs ys} = ((\text{mux xs ys})_{2n} == \text{ys}_n \text{ if } n \leq \text{length ys}) \]
Informal specification of mux

- $\text{mux} \ [x_1, x_2, \ldots, x_n] \ [y_1, y_2, \ldots, y_n] = [x_1, y_1, x_2, y_2, \ldots, x_n, y_n]$
- Note: append extra elements in either sequence to the end

Suppose we've defined mux in a program and want to test it

- Example test: $\text{mux} \ [1, 2, 3] \ [4, 5, 6, 7] == [1, 4, 2, 5, 3, 6, 7]$

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  - $\text{muxTest} \ xs \ ys = ((\text{mux} \ xs \ ys)_{2n+1} == xs_n \text{ if } n \leq \text{length} \ xs)$
  - $\text{muxTest} \ xs \ ys = ((\text{mux} \ xs \ ys)_{2n} == ys_n \text{ if } n \leq \text{length} \ ys)$

Let's think about a test that is easier to generalize

- $\text{mux} \ [x_1, x_2, \ldots, x_{n+1}] \ [y_1, y_2, \ldots, y_{m+1}]$
  - $= [x_1, y_1] ++ (\text{mux} \ [x_2, \ldots, x_{n+1}] \ [y_2, \ldots, y_{m+1}])$
Informal specification of mux
- \( \text{mux} \left[ x_1, x_2, \ldots, x_n \right] \left[ y_1, y_2, \ldots, y_n \right] = [x_1, y_1, x_2, y_2, \ldots, x_n, y_n] \)
- Note: append extra elements in either sequence to the end

Suppose we've defined mux in a program and want to test it
- Example test: \( \text{mux} \left[ 1, 2, 3 \right] \left[ 4, 5, 6, 7 \right] = [1, 4, 2, 5, 3, 6, 7] \)

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- The following tests are wrong with lists of differing lengths
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Let's think about a test that is easier to generalize
- \( \text{mux} \left[ x_1, x_2, \ldots, x_{n+1} \right] \left[ y_1, y_2, \ldots, y_{m+1} \right] = [x_1, y_1] ++ (\text{mux} \left[ x_2, \ldots, x_{n+1} \right] \left[ y_2, \ldots, y_{m+1} \right]) \)

We can correctly generalize that test as follows
- \( \text{muxTest} \ (x : \text{xs}) (y : \text{ys}) = ((\text{mux} \ (x : \text{xs}) (y : \text{ys})) = ([x, y] ++ (\text{mux} \ \text{xs} \ \text{ys}))) \)

Notation
- \( x : xs \leftarrow \text{shorthand for } [x] ++ xs \)

We can use \text{muxTest} to run thousands of tests, automatically
**Multiplexor – Defining Properties**

- Let’s think about a test that is easier to generalize
  - $\text{mux} [x_1, x_2, \ldots x_{n+1}] [y_1, y_2, \ldots y_{m+1}]$
    
    $= [x_1, y_1] ++ (\text{mux} [x_2, \ldots x_{n+1}] [y_2, \ldots y_{m+1}])$

- We can correctly generalize that test as follows
  - $\text{muxTest} (x : xs) (y : ys) = ((\text{mux} (x : xs) (y : ys)) == ([x, y] ++ (\text{mux} xs ys)))$

- Notation
  - $x : xs$ ← shorthand for $[x] ++ xs$

- We can use $\text{muxTest}$ to run thousands of tests, automatically

- **How about some corner cases?**
  - $\text{mux} xs [\ ] = xs$
  - $\text{mux} [\ ] (y : ys) = (y : ys)$
Multiplexor – Defining Properties

- Let’s think about a test that is easier to generalize
  - \( \text{mux } [x_1, x_2, \ldots x_{n+1}] [y_1, y_2, \ldots y_{m+1}] = [x_1, y_1] ++ (\text{mux } [x_2, \ldots x_{n+1}] [y_2, \ldots y_{m+1}]) \)

- We can correctly generalize that test as follows
  - \( \text{muxTest } (x : xs) (y : ys) = ((\text{mux } (x : xs) (y : ys)) == ([x, y] ++ (\text{mux } xs ys))) \)

- Notation
  - \( x : xs \) ← shorthand for \([x] ++ xs\)

- We can use muxTest to run thousands of tests, automatically

- How about some corner cases?
  - \( \text{mux } xs [ ] = xs \)
  - \( \text{mux } [ ] (y : ys) = (y : ys) \)

- Now we have three simple properties that mux must satisfy
  - \( \text{mux } (x : xs) (y : ys) = [x, y] ++ (\text{mux } xs ys) \)
  - \( \text{mux } xs [ ] = [ ] \)
  - \( \text{mux } [ ] (y : ys) = (y : ys) \)
Let’s think about a test that is easier to generalize

\[ \text{mux} \left[ x_1, x_2, \ldots x_{n+1} \right] \left[ y_1, y_2, \ldots y_{m+1} \right] = [x_1, y_1] ++ (\text{mux} \left[ x_2, \ldots x_{n+1} \right] \left[ y_2, \ldots y_{m+1} \right]) \]

We can correctly generalize that test as follows

\[ \text{muxTest} (x : xs) (y : ys) = ((\text{mux} (x : xs) (y : ys)) == ([x, y] ++ (\text{mux xs ys}))) \]

Notation

- \( x : xs \) ← shorthand for \([x] ++ xs\)

We can use \( \text{muxTest} \) to run thousands of tests, automatically

How about some corner cases?

- \( \text{mux} \  xs \ [\ ] = xs \)
- \( \text{mux} \ [\ ] (y : ys) = (y : ys) \)

Now we have three simple properties that \( \text{mux} \) must satisfy

- \( \text{mux} \ (x : xs) (y : ys) = [x, y] ++ (\text{mux} \ xs \ ys) \)
- \( \text{mux} \ xs \ [\ ] = [\ ] \)
- \( \text{mux} \ [\ ] (y : ys) = (y : ys) \)

Surprise! All properties of \( \text{mux} \) derive from these

- That is, these properties form an \textbf{inductive definition} of \( \text{mux} \)
How to Define Functions

- Now we have three simple properties that mux must satisfy
  - \( \text{mux} \ (x : xs) \ (y : ys) = [x, y] ++ (\text{mux} \ xs \ ys) \)
  - \( \text{mux} \ xs \ [\ ] = xs \)
  - \( \text{mux} \ [\ ] \ (y : ys) = (y : ys) \)

- Surprise! All properties of mux derive from these
  - In other words, these simple properties define mux

How to write inductive definitions

1. Write simple equations that the desired function must satisfy
2. Make sure there is an equation for each expected form of input
3. Make sure the input for each inductive reference is closer to a non-inductive case than the input form covered by the equation

- Writing reliable programs is as simple as that!
Axiomatic vs Derived Properties

Now we have three simple properties that `mux` must satisfy:

- `mux (x : xs) (y : ys) = [x, y] ++ (mux xs ys)` \{Axiom M₁\}
- `mux xs [] = xs` \{Axiom M₀.₂\}
- `mux [] (y : ys) = (y : ys)` \{Axiom M₀.₁\}

Surprise! All properties of `mux` derive from these.

In other words, these simple properties define `mux`.

How to write inductive definitions

1. Write simple equations that the desired function must satisfy
2. Make sure there is an equation for each expected form of input
3. Make sure the input for each inductive reference is closer to a non-inductive case than the input form covered by the equation

Writing reliable programs is as simple as that!

- Even better, you can derive additional properties by induction
- Such as, for example, these properties
  - \{P₁\} if (elem x xs) then (elem x (mux xs ys))
  - \{P₂\} if (elem y ys) then (elem y (mux xs ys))
  - \{P₃\} if (elem z (mux xs ys)) then ((elem z xs) || (elem z ys))

Notation

- `(elem x xs)` is True iff `x` is an element of the sequence `xs`
Derived Properties are Theorems

- Now we have three simple properties that `mux` must satisfy
  - \[ \text{mux} (x : xs) (y : ys) = [x, y] ++ (\text{mux} xs ys) \] \{Axiom M₁\}  
  - \[ \text{mux} xs \ [\ ] = xs \] \{Axiom M₀₂\}  
  - \[ \text{mux} \ [\ ] (y : ys) = (y : ys) \] \{Axiom M₀₁\}  

- Surprise! All properties of `mux` derive from these
  - In other words, these two simple properties define `mux`

- How to write inductive definitions
  1. Write simple equations that the desired function must satisfy  
  2. Make sure there is an equation for each expected form of input  
  3. Make sure the input for each inductive reference is closer to a non-inductive case than the input form covered by the equation

- Writing reliable programs is as simple as that!
  - Even better, you can derive additional properties by induction  
  - Such as, for example, these properties
    - \{P₁\} if (elem x xs) then (elem x (mux xs ys))  
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- Notation
  - `(elem x xs)` is True iff `x` is an element of the sequence `xs`
Property Derived by Induction

Now we have three simple properties that mux must satisfy

- \( \text{mux} (x : xs) (y : ys) = [x, y] ++ (\text{mux} xs ys) \) \{Axiom M_1\}
- \( \text{mux} xs [ ] = xs \) \{Axiom M_{0.2}\}
- \( \text{mux} [ ] (y : ys) = (y : ys) \) \{Axiom M_{0.1}\}
- \( \text{elem} x [ ] = \text{False} \) \{Axiom E_o\}
- \( \text{elem} y (x : xs) = (y == x) \lor (\text{elem} y xs) \) \{Axiom E_1\}

Theorem \( P_1 \): if \( \text{elem} x xs \) then \( \text{elem} x (\text{mux} xs ys) \)

that is, \( (\text{elem} x xs) \rightarrow (\text{elem} x (\text{mux} xs ys)) \) \{as Boolean formula\}

- Base case
  \( (\text{elem} x [ ]) \rightarrow (\text{elem} x (\text{mux} [ ] ys)) \)
  \( = \text{False} \rightarrow (\text{elem} x (\text{mux} [ ] ys)) \) \{Axiom E_o\}
  \( = \text{True} \) \{Boolean algebra\}
Property Derived by Induction

- Now we have three simple properties that mux must satisfy:
  - \( \text{mux} (x : xs) (y : ys) = [x, y] ++ \text{mux} \text{ xs} \text{ ys} \) \{Axiom \( M_1 \}\}
  - \( \text{mux} \text{ xs} [\ ] = \text{xs} \) \{Axiom \( M_{0,2} \}\}
  - \( \text{mux} [\ ] (y : ys) = (y : ys) \) \{Axiom \( M_{0,1} \}\}
  - \( \text{elem} x [\ ] = \text{False} \) \{Axiom \( E_0 \)\}
  - \( \text{elem} y (x : xs) = (y == x) \| (\text{elem} y \text{ xs}) \) \{Axiom \( E_1 \)\}

- Theorem \( P_1 \): if \((\text{elem} x \text{ xs})\) then \((\text{elem} x \text{ (mux xs ys)})\)
  that is, \((\text{elem} x \text{ xs}) \rightarrow (\text{elem} x \text{ (mux xs ys)})\) \{as Boolean formula\}

- Inductive case when \( ys \neq [\ ] \) Note: proof is simpler when \( ys = [\ ] \)

\[
\begin{align*}
\text{(elem} x (x_1 : xs)) & \rightarrow (\text{elem} x (\text{mux} (x_1 : xs) (z: zs))) \quad \{\text{rewrite non-empty} \ ys \text{ as} \ (z: zs)\} \\
= ((x==x_1) \| (\text{elem} x \text{ xs})) & \rightarrow (\text{elem} x ([x_1, z] ++ (\text{mux} \text{ xs} \text{ zs}))) \quad \{\text{Axioms} \ E_1 \text{ and} \ M_1\}
\end{align*}
\]

\[
\begin{align*}
= ((x==x_1) \| (\text{elem} x \text{ xs})) & \rightarrow (\text{elem} x (x_1: z: (\text{mux} \text{ xs} \text{ zs}))) \quad \{\text{shorthand for} \ ++\}
\end{align*}
\]

\[
\begin{align*}
= ((x==x_1) \| (\text{elem} x \text{ xs})) & \rightarrow ((x==x_1) \| (x==z) \| (\text{elem} x (\text{mux} \text{ xs} \text{ zs}))) \quad \{\Ax \ E_1\}
\end{align*}
\]

\[
\begin{align*}
= ((x==x_1) \rightarrow ((x==x_1) \| (x==z) \| (\text{elem} x (\text{mux} \text{ xs} \text{ zs})))) & \quad \&\& \\
(\text{elem} x \text{ xs}) & \rightarrow ((x==x_1) \| (x==z) \| (\text{elem} x (\text{mux} \text{ xs} \text{ zs}))) \quad \{\text{Boolean algebra}\}
\end{align*}
\]

\[
= \text{True} \&\& \\
(\text{elem} x \text{ xs}) & \rightarrow ((x==x_1) \| (x==z) \| (\text{elem} x (\text{mux} \text{ xs} \text{ zs}))) \\
= \text{True} \quad \{\text{induction hypothesis, along with some Boolean algebra}\}
\]
Using Logic to Ensure Reliable Software

- Pencil and paper proofs are just for practice
  - Enables students to learn methods of reasoning
  - Logic course at Oklahoma Univ includes hundreds of examples
  - Proofs follow the rigorous style of mux example presented here

- Real-life software engineering requires mechanized logic
  - Getting all the details right in proofs isn't humanly possible
  - Proofs with incorrect details are not proofs
  - Non-proofs don't guarantee software properties
  - Therefore, paper-and-pencil proofs provide no guarantees

- Mechanized logic verifies all the details, down to bit level
  - So, it's feasible to know software properties for certain
  - Also, mechanized logic can partially automate proofs
  - This makes mechanized logic useful in the practice of SE

- Practical mechanized logic for industry: ACL2
  - SE course at Oklahoma Univ uses ACL2 with [Proof Pad](http://www.cs.ou.edu/~rlpage/SEcollab/EngrSwJFP.pdf)
The End