Student Name: ______________________ Student ID # ______________________

UOSA Statement of Academic Integrity

On my honor I affirm that I have neither given nor received inappropriate aid in the completion of this exercise.

Signature: ______________________ Date: ______________________

Notes Regarding this Examination

Open Book(s) You may consult any printed textbooks in your immediate possession during the course of this examination.

Open Notes You may consult any printed notes in your immediate possession during the course of this examination.

No Electronic Devices Permitted You may not use any electronic devices during the course of this examination, including but not limited to calculators, computers, and cellular phones. All electronic devices in the student's possession must be turned off and placed out of sight (for example, in the student's own pocket or backpack) for the duration of the examination.

Violations Copying another's work, or possession of electronic computing or communication devices in the testing area, is cheating and grounds for penalties in accordance with school policies.

Question E1-1: Well-Formed Formulas (10 points)

Consider the following well-formed formula: $(a \rightarrow b) \leftrightarrow (a \veebar b)$

A. Is it satisfiable? *Justify* your answer.

B. Is it tautologous? *Justify* your answer.

C. Draw its circuit diagram.

Question E1-2: Well-Formed Formulas (10 points)

Consider the following well-formed formula: $((a \rightarrow b) \leftarrow c) \vee ((a \leftarrow b) \rightarrow c)$

A. Is it a contradiction? *Justify* your answer.

B. Is it tautologous? *Justify* your answer.

C. Draw its circuit diagram.

Question E1-3: Natural Deduction (10 points)

Consider the following partial theorem and corresponding partial proof:

Theorem: $?, b \rightarrow c \vdash \neg a \rightarrow c$

Proof:

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{a \lor b \qquad ?}{b}\,\{?\} \qquad ?}{c}\,\{?\} \qquad ?}{\text{False}}\,\{?\}}{c}\,\{?\}}{?}\,\{?\}$$

Rewrite the theorem and the proof, filling in the missing parts (marked with '?' in the theorem and proof). Mark all assumptions (if any) that will be discharged and indicate the rule citations that cause discharges.

Question E1-4: Natural Deduction (10 points)

Prove the following theorem using natural deduction: $\neg a \lor b, a \lor c, \neg c \vdash b$

Question E1-5: Natural Deduction (20 points)

Prove the following theorem using natural deduction: $(a \wedge b \wedge c) \vee d \vdash (a \vee d) \wedge (b \vee d) \wedge (c \vee d)$

Question E1-6: Equational Reasoning (20 points)

Prove the following using equational reasoning: $(\neg c \wedge (\neg a \vee (b \vee c))) = ((a \rightarrow b) \wedge \neg c)$

Question E1-7: Equational Reasoning (20 points)

Prove the following using equational reasoning: $a = ((a \wedge c) \vee (a \wedge (c \rightarrow a)))$

Question E2-1: Quantified Natural Deduction (10 points)

Prove the following theorem using natural deduction:

$$(\forall x.((G(x) \to H(x)) \land (F(x) \to G(x)))) \vdash (\forall x.(F(x) \to H(x)))$$

Question E2-2: Quantified Equational Reasoning (10 points)

Prove the following equation using equational reasoning:

$$(\forall x.(F(x) \vee G(x))) = (\neg \exists x.(\neg F(x) \wedge \neg G(x)))$$

Question E2-3: Sets (10 points)

Prove the following equation involving sets A, B, and C:

$$((B \cap A) - (B \cap C)) = ((A \cap B) - C)$$

Question E2-4: Circuit Minimization using Karnaugh Maps (10 points)

Consider the following Boolean function F.

a	b	c	d	F(a,b,c,d)	minterm
0	0	0	0	0	
0	0	0	1	0	
0	0	1	0	0	
0	0	1	1	1	
0	1	0	0	0	
0	1	0	1	0	
0	1	1	0	0	
0	1	1	1	0	
1	0	0	0	1	
1	0	0	1	1	
1	0	1	0	1	
1	0	1	1	x	
1	1	0	0	1	
1	1	0	1	1	
1	1	1	0	0	
1	1	1	1	x	

A. Write the minterms of F in the column above for the given function.

B. Create a Karnaugh map for F.

C. Use the Karnaugh map to find a minimum sum-of-products representation of F, give that representation, and show how you arrived at the representation from the map.

Question E2-5: Induction (20 points)

Theorem {++tail}: $\forall$n.([x_1, x_2, ... x_n] ++ [x_{n+1}] = [x_1, x_2, ... x_n, x_{n+1}])
Prove Theorem {++tail} by induction.

Question E2-6: Induction Redux (20 points)

Consider the following type definition and axioms:

reverse::[a] $\rightarrow$ [a]
reverse [] = [] $\{$rev[]$\}$
reverse (x:xs) = (reverse xs) ++ [x] $\{$rev:$\}$

Theorem $\{$rev thm$\}$: $\forall$n.(reverse [d_1, d_2, ... d_{n-1}, d_n] = [d_n, d_{n-1}, ... d_2, d_1])
Prove Theorem $\{$rev thm$\}$ by induction.
For this proof, you may use Theorem $\{$++tail$\}$ from Question E2-5, even if you have not proven it.

Question E2-7: More Induction Redux (20 points)

Consider the following type definitions and axioms:

one:: a $\rightarrow$ Int

one a = 1 {one}

Prove that for all xs where xs is a sequence: ((foldr (+) 0 (map one xs)) = (length xs))

Question E3-1: Induction (20 points)

Theorem {tail length}: $\forall$xs.(length (xs ++ [y]) = (1 + length xs)) where xs is a sequence.
Prove Theorem {tail length} using induction on length xs.

Question E3-2: Strong Induction (30 points)

Consider the following type definitions and axioms:

tri:: [a] $\rightarrow$ ([a], [a], [a])

tri [] = ([], [], []) $\{$tri[]$\}$

tri [a] = ([a], [], []) $\{$tri[a]$\}$

tri [a, b] = ([a], [b], []) $\{$tri[a, b]$\}$

tri (a : (b : (c : xs))) = (a : as, b : bs, c : cs) where (as, bs, cs) = tri xs $\{$tri:::$\}$

Prove: $\forall$n.((as, bs, cs) = tri[x_1, x_2, ... x_n]) $\rightarrow$ (((length as) + (length bs) + (length cs)) = n)

(Additional space to complete Question E3-2.)

Question E3-3: Computation Time (20 points)

Using the type definitions and axioms from Question E3-2, prove:

$\forall$n.((as, bs, cs) = tri[x_1, x_2, ... x_n]) $\rightarrow$ (tri[x_1, x_2, ... x_n] terminates)

Question E3-4: Numerical Systems (30 points)

Prove using induction on length ds for all ds where ds is a sequence:

(length (trim ds)) $\leq$ (length ds)

For this proof, you may use Theorems {++tail}, {rev thm}, and/or {tail length} from prior questions on this exam, even if you have not proven them.

(Additional space to complete Question E3-4.)

Question E3-5: Adders (10 points)

Use the following theorem to design a circuit that, given a w-bit binary numeral, delivers the w-bit 2's complement of the numeral.

Theorem {2s trick} (from Lecture 21):

$\forall$w.$\forall$n$\in$I(w).(((word w n) = [b_0, b_1, ... b_{w-1}]) $\rightarrow$

((word w (-n)) = word w (1+num[1-b_0, 1-b_1, ... 1-b_{w-1}])))

Question E3-6: Tree Induction (40 points)

Consider the following type definition and axioms:

size:: (SearchTree key dat s, Num n) $\Rightarrow$ s $\rightarrow$ n
size Nub = 0 {size Nub}
size (Cel k d left right) = 1 + (size left) + (size right) {size Cel}

Prove using Tree Induction: $\forall h \in \mathcal{N}. \exists s.(((\text{height } s)=h) \land (\text{size } s = (2\hat{\ }h\text{-}1)))$ where s is a SearchTree.

(Additional space to complete Question E3-6.)