Control Theory: A Basis for Reactive Behavior

Problem Statement

"Find control law" rather than "find path" problem.

Design a control surface that reacts to incremental sensor information and preserves formal notions of stability and convergence.

stability — the property of a regulated system that suppresses perturbations and returns the system to the equilibrium state from a bounded neighborhood containing the equilibrium state.

convergence — the property of a regulated system that causes it to approach the equilibrium state asymptotically and come to rest.

Navigation Function

$\vec{V}(x , t)$

$\vec{x}$: state vector

t : time

$$\frac{dV}{dt} = \underbrace{\frac{dV}{d\vec{x}}} \; \underbrace{\frac{d\vec{x}}{dt}}$$

performance criteria **system dynamics**

- obstacle avoidance
- active sensing objective
- grasp objective
- etc.

Analytic Stability — Lyapunov's Direct Method

Definition 0.1 *The origin of the state space is stable if there exists a region, $S(r)$, such that states which start within $S(r)$ remain within $S(r)$.*

Definition 0.2 *Systems which satisfy Definition 0.1 are asymptotically stable if as $t \to \infty$, the system state approaches the origin of the state space.*

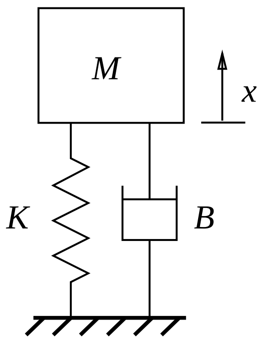
Analytic Stability

Definition 0.3 ***Lyapunov's Second Method*** *If the function, $V(\vec{x}, t)$, exists such that:*

$$
\begin{array}{lcll}
V(\vec{0}, t) & = & 0, \; and & \\
V(\vec{x}, t) & > & 0, \; for \; x \neq 0 & (positive \; definite), \; and \\
\partial V / \partial t & < & 0 & (negative \; definite),
\end{array}
$$

then, the state space described by V is asymptotically stable in the neighborhood of the origin. If a system is stable, then there is a suitable Lyapunov function. If however, a particular Lyapunov function does not satisfy these criteria, it is not necessarily true that this system is unstable.

EXAMPLE: spring-mass-damper



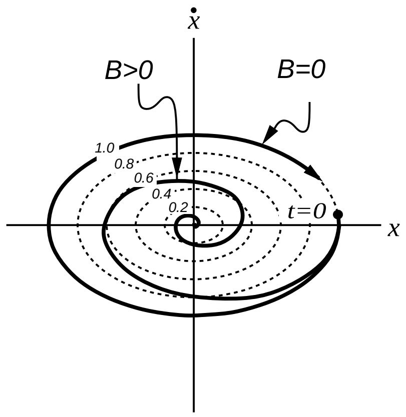
system dynamics:

$\ddot{x} + \frac{B}{M}\dot{x} + \frac{K}{M}x = 0$

quadratic Lyapunov function:

$L = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}Kx^2$ *positive definite*

$$\begin{aligned}\frac{dL}{dt} &= M\dot{x}\ddot{x} + Kx\dot{x} = M\dot{x}(-\frac{B}{M}\dot{x} - \frac{K}{M}x) + Kx\dot{x} \\ &= -B\dot{x}^2 \qquad\qquad negative\ definite\end{aligned}$$



EXAMPLE: population dynamics

system dynamics:

$x_1 = \#$ males $\qquad \dot{x}_1 = -x_1 + \alpha x_1 x_2 = x_1(\alpha x_2 - 1)$

$x_2 = \#$ females $\qquad \dot{x}_2 = -x_2 + \beta x_1 x_2 = x_2(\beta x_1 - 1)$

equilibrium points: $\dot{\vec{x}} = \vec{0}$

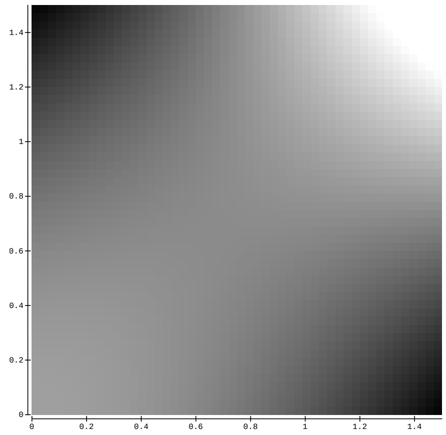
(a) $x_1 = x_2 = 0$
(b) $x_1 = \frac{1}{\beta}, x_2 = \frac{1}{\alpha}$

Lyapunov function:

$$\left.\begin{array}{l} V(\vec{0}, t) = 0 \\ V(\vec{x}, t) > 0 \end{array}\right\} \quad \text{choose } V(\vec{x}, t) = x_1^2 + x_2^2$$

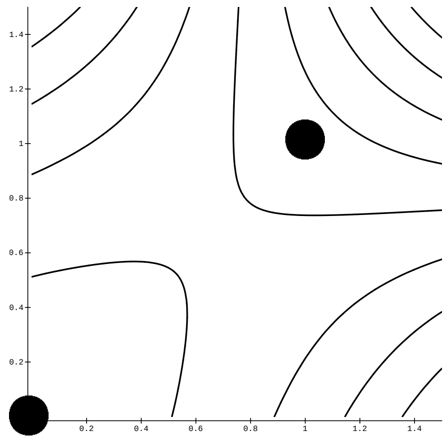
$$\begin{aligned} \frac{\partial V}{\partial t} &= 2x_1\dot{x}_1 + 2x_2\dot{x}_2 \\ &= 2x_1^2(\alpha x_2 - 1) + 2x_2^2(\beta x_1 - 1) \leq 0 \end{aligned}$$

EXAMPLE: population dynamics



level curves
in dV/dt function

x_2
females



x_1
males

