Spatial properties of the image function

Dirac delta operator:

$$\delta(x-\xi) = \begin{cases} \infty & x = \xi \\ 0 & otherwise \end{cases}$$

This function has the following properties:

$$\int_{-\epsilon}^{\epsilon} \delta(x)dx \;=\; 1 \quad for \; \epsilon > 0$$

$$\int_{-\infty}^{\infty} F(\xi)\delta(x-\xi)d\xi \;=\; F(x) \qquad \textbf{Sifting Property}$$

Spatial properties of the image function

Fourier Transform

$$\mathcal{F}[f(x,y)] = F(u,v) = \iint_{-\infty}^{\infty} f(x,y) exp\{-i(2\pi(ux+vy)\} dxdy$$

where u is the spatial frequency (in $cycles/pixel$ perhaps, so that when x is specified in pixels, $(2\pi ux)$ is in radians, and $i = \sqrt{-1}$.

$$\mathcal{F}^{-1}[F(u,v)] = f(x,y) = \iint_{-\infty}^{\infty} F(u,v) exp\{i(2\pi(ux+vy))\} dudv$$

Shift and Convolution Theorems

Shift Theorem:

$$
\begin{aligned}
\mathcal{F}[f(x)] &= \int_{-\infty}^{\infty} f(x) exp\{-i(2\pi ux)\} dx, \ then \\
\mathcal{F}[f(x-a)] &= \int_{-\infty}^{\infty} f(x-a) exp\{-i(2\pi ux)\} dx, \ then \\
&= \int_{-\infty}^{\infty} f(x') exp\{-i(2\pi u(x'+a))\} dx', \ and \\
&= exp\{-i(2\pi ua)\} \int_{-\infty}^{\infty} f(x') exp\{-i(2\pi ux')\} dx',
\end{aligned}
$$

so that

$$
\mathcal{F}[f(x-a)] = exp\{-i(2\pi ua)\}\mathcal{F}(f(x))
$$

Shift and Convolution Theorems

Convolution Theorem:

The convolution of two functions $f(x)$ and $g(x)$:

$$f(x) * g(x) = h(x) = \int_{-\infty}^{\infty} f(\alpha) g(x-\alpha) d\alpha$$

where α an integration variable.

$$\begin{aligned}
\mathcal{F}[f(x) * g(x)] &= \mathcal{F}[h(x)] \\
&= \mathcal{F}\left[\int_\alpha f(\alpha) g(x-\alpha) d\alpha\right] \\
&= \int_x \left[\int_\alpha f(\alpha) g(x-\alpha) d\alpha\right] exp\{-i2\pi ux\} dx \\
&= \int_\alpha f(\alpha) \left[\int_x g(x-\alpha) exp\{-i(2\pi ux)\} dx\right] d\alpha
\end{aligned}$$

and by the Shift theorem,

$$= \int_\alpha f(\alpha) exp\{-i(2\pi u\alpha)\} d\alpha \int_x g(x) exp\{-i(2\pi ux)\} dx,$$

therefore,

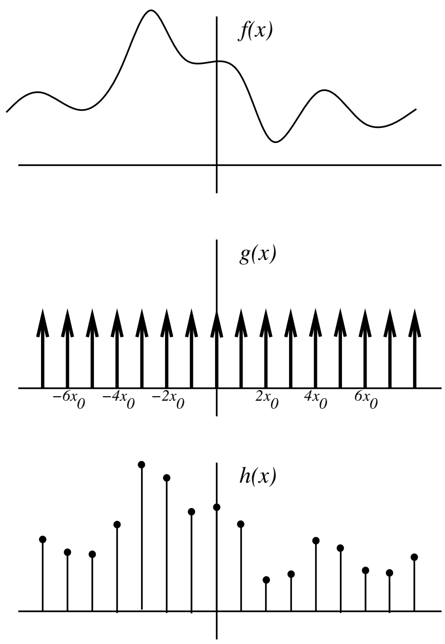
$$\mathcal{F}[f(x) * g(x)] = F(u)G(u)$$

therefore, spatial convolution $\Longleftrightarrow$ frequency domain multiplication

Sampling Theorem - Aliasing

A continuous spatial function, $f(x)$ is *sampled* by computing the product of $f(x)$ and $g(x)$, an infinite sequence of dirac delta operators.

$$\begin{aligned} h(x) &= f(x)\sum_n \delta(x - nx_0) \\ &= \sum_n f(nx_0)\delta(x - nx_0) \end{aligned}$$



By the convolution theorem, we know that the product of these two spatial functions is equivalent to the convolution of their Fourier transform pairs.

$$\begin{aligned} f(x) &\xrightarrow{\mathcal{F}} F(u) \\ \sum_n \delta(x - nx_0) &\xrightarrow{\mathcal{F}} \frac{1}{x_0}\sum_n \delta(u - \frac{n}{x_0}) = G(u),\ \ and \\ H(u) &= F(u) * G(u) \end{aligned}$$

Sampling Theorem - Aliasing (cont.)

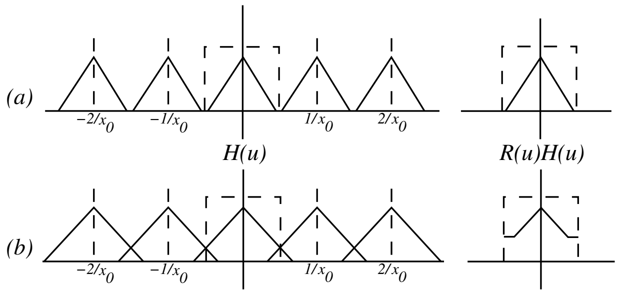
$$H(u) = F(u) * G(u)$$

We may write the function $H(u)$ in terms of $F(u)$:

$$\begin{aligned}
H(u) &= F(u)*\left[\frac{1}{x_0}\sum_n \delta(u-\frac{n}{x_0})\right] \\
&= \int_{-\infty}^{\infty} F(\alpha)G(u-\alpha) \\
&= \int_{-\infty}^{\infty} F(\alpha)\left[\frac{1}{x_0}\sum_n \delta(u-\alpha-\frac{n}{x_0})d\alpha\right] \\
&= \frac{1}{x_0}\int_{-\infty}^{\infty}\sum_n F(u-\frac{n}{x_0})\delta(u-\alpha-\frac{n}{x_0})d\alpha \\
&= \frac{1}{x_0}\sum_n F(u-\frac{n}{x_0})\int_{-\infty}^{\infty}\delta(u-\alpha-\frac{n}{x_0})d\alpha \\
H(u) &= \frac{1}{x_0}\sum_n F(u-\frac{n}{x_0})
\end{aligned}$$

Therefore, the frequency spectrum of the sampled image consists of duplicates of the spectrum of the original image distributed at $1/x_0$ frequency intervals.

Aliasing (cont.)



$R(u)$ is a frequency domain bandpass filter

$$R(u) = \begin{array}{l} 1 \;\; if \;\; |u| < 1/(2x_0), \\ 0 \;\; otherwise \end{array}$$

Aliasing: When replicated spectra interfere, the crosstalk introduces energy at relatively high frequencies changing the appearance of the reconstructed image.

The Sampling Theorem: *if the image contains no frequency components greater than one half the sampling frequency, then the continuous image is faithfully represented in the sampled image.*