

Concurrent Communication in High-Speed Wide Area Networks

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Abstract—A performance metric called *receptivity* is introduced for quantifying the degree of concurrent communication possible in high-speed wide area networks (WAN's). Given a stochastic demand pattern model, *receptivity* is defined to be the probability that all requested connections can be established concurrently. Because calculation of the exact value of *receptivity* is shown to (generally) have an exponential complexity, an analytic estimate for its value is derived. The derived estimate is dependent on network parameters such as the number of links, link capacity values, and a weighted hop distance metric (which depends on the topological structure of the network and its relationship to parameter values of the stochastic model for the demand patterns). The derived estimate for the proposed metric compares reasonably well with simulated values for several asymmetric topological structures ranging from planar meshes to random graphs. The utility of the estimate is twofold. First, it can be computed quickly, i.e., in polynomial time. Second, its simple analytic form provides the network architect with insight into some of the inherent limitations and consequences associated with topological design choices for high-speed WAN's.

Index Terms—Wide area networks, gigabit networks, performance evaluation, performance estimation, concurrent communication, topology design

I. INTRODUCTION

A. Motivation

IN WIDE AREA networks (WAN's), the effectiveness of reactive congestion avoidance techniques (e.g., admission control) diminishes as the transmission rates of the links are increased [14], [15]. To understand why this is true, note first that the propagation time associated with a communication link is a fixed constant (because the speed of light is constant). So, as link transmission rates are increased, the ratio of propagation delay time to bit transmission time increases. As a result, by the time state information is acquired for making a reactive control decision, the actual state of the network may have changed significantly. In other words, as link transmission rates are increased, the dynamics for the network's state become faster; because link propagation delay is constant, however, acquired state information becomes relatively outdated when transmission rates are increased. For this reason, understanding how the topological structure of a network impacts its effectiveness in processing communication patterns—under the assumption that the network has limited reactive control mechanisms, or none at all, in

place—becomes critical for current and future high-speed (e.g., gigabits/s) WAN's.

In the past, because of the effectiveness of reactive control techniques (when link capacities were on the order of megabits/s and below), topological design choices for WAN's were largely driven by economics. In particular, the effectiveness of reactive control enabled network architects to make engineering trade-offs between the construction cost of a network topology and the associated network performance. For instance, whereas the most economical topology choice for a given aggregate bandwidth might not closely match the expected traffic demand pattern, it could nevertheless be chosen because the employment of reactive control techniques enabled even the most "miss-matched" topologies to perform at least moderately well.

There have been some formal techniques reported in the literature for optimally interconnecting geographically distributed networks [1], [3], [4], [5]. As a representative example, in [3], an iterative algorithm is proposed for optimal topology design that is based on solving a constrained optimization problem. The overall objective of this particular technique, given estimates for the expected communication pattern, is to interconnect a geographically distributed set of nodes to minimize the expected average delay through the network. Various aspects of capital investment (i.e., the number, physical length, and bandwidth of the interconnecting communication links) are incorporated as constraints in the formulation of this optimization problem. Although the design technique such as the one introduced in [3] is interesting from a theoretical point of view, in practice, network topologies tend to evolve over time, with design choices being made incrementally, not all at once. Thus, some intuitive guidelines may be useful to the architect for making incremental topological changes.

B. Goal of the Paper

In this paper, the goal is to quantify how topological structure affects network performance, in the absence of any reactive network control mechanism. As stated in the previous subsection, the effectiveness of reactive control techniques in improving performance for current and future high-speed networks will diminish as link rates continue to increase.

Although it is inherently difficult to react dynamically (in real-time) to changing demands in high-speed WAN's, it may be possible to predict trends associated with the communication demands on an hourly or daily time-scale by using probabilistic time series techniques. Electric utility

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companies have used such techniques for decades to aid in matching produced power to the demand, and for matching the topological structure of the transmission network to the expected demand patterns.

In some optical high-speed WAN's, the capability exists to change the virtual topological structure of the network (electronically) by defining the virtual channels between origin–destination pairs. An example of a an optical network capable of emulating various point-to-point networks using virtual channels has been proposed and built [16]. Of course, the time-scale for changing the virtual topological structure of such a network is not fast enough to react dynamically to changes in demand patterns. Assuming that the demands have relatively slowly varying trends over longer periods of time (in a statistical sense), however, the virtual topology of such networks could be changed to best match the expected demand pattern. The results presented in this paper provide simple guidelines to aid in the selection of effective topological structures.

The performance of a high-speed WAN is characterized here based on its ability to accommodate concurrent communication requests between distinct origin–destination pairs. The primary focus is to investigate how the WAN's topological structure affects its ability to accommodate concurrent communication. To this end, a graph-theoretic result is first derived that provides an upper bound for the number of concurrently communicating origin–destination pairs that can be accommodated by a given network. The bound is dependent on topological attributes such as weighted hop distances, number and capacity of links, and network diameter. The bound is also dependent on parameter values of a stochastic model for the origin–destination communication demands. Through simulation studies, the bound is shown to be asymptotically tight for several types of asymmetric topological structures. The bound is then used as a cornerstone for developing a simple formula to estimate the receptivity of general WAN's (receptivity is defined as the probability that a given network will be able to accommodate all concurrent communication requests). This estimate for the receptivity of a network is useful on two counts. First, it can be computed quickly (relative to the amount of computation time generally required to determine the exact value of receptivity). Second, its simple analytic form naturally exposes important trade-offs and limitations associated with certain network design choices—in terms of how the values of these design parameters impact the network's measure of receptivity.

C. Related Work

The concept of modeling communication networks as link capacitated graphs (as is done in the present paper) is not new. In fact, there are several classic results that have been derived through the years. Perhaps the most notable is the so-called Min-Cut/Max-Flow Theorem associated with the maximum flow problem [12, p. 178]. In the maximum flow problem the objective is to determine the maximum amount of flow possible between a single origin–destination pair. There are other classic results for determining whether a feasible routing exists for a given set of requesting origin–destination pairs [12, pp. 335–345].

Although the existence of feasible routings is part of what is considered in the present paper, the main concern is in estimating the probability that a feasible routing solution will exist. To our knowledge, the problem formulation introduced in the present paper is new.

D. Organization of the Paper

In Section II, a mathematical network model and the associated notations are introduced. Section III contains the formal definition of network receptivity, along with some simple (special case) examples for which receptivity can be easily computed. Section IV addresses the generic difficulty in computing the exact value of receptivity. In Section V, an expression is derived that provides for an “easy-to-compute” estimate of receptivity. Also, this result is useful in that its simple form provides an indication of some of the inherent limitations and trade-offs associated with maximizing receptivity under practical design constraints. Finally, values from the estimate of receptivity are compared with simulated values of receptivity.

II. THE NETWORK MODEL

A. Preliminaries

An n -node WAN is modeled as a link capacitated directed graph $G = (\mathcal{N}, \mathcal{L}, \mathcal{C})$, where $\mathcal{N} = \{1, \dots, n\}$ denotes the set of nodes, $(i, j) \in \mathcal{L}$ is the set of interconnecting links, and $C_{ij} \in \mathcal{C}$ is the associated set of link capacities. The physical interpretation of the link capacity is defined as follows.

Definition: The capacity of link (i, j) , denoted as C_{ij} , is defined as the maximum possible bit-rate of incoming data that node i can receive—possibly simultaneously from any or all neighboring nodes—and successfully transmit to node j in finite time, via link (i, j) .

Obviously, the above definition of a link's capacity hides the details of how the transmissions are accomplished, e.g., whether queues are used at each outgoing link, and what, if any, type of link control scheme is employed. For the purposes of this paper, however, the above model captures the essence of high-speed WAN links.

Also associated with each link is a flow rate, defined below.

Definition: The flow on link $(i, j) \in \mathcal{L}$ at time t , denoted as $f_{ij}(t)$, is defined as the total rate at which data are received at node i and destined for link (i, j) , at time t .

The next definition provides notation for describing the set of all possible origin–destination pairs.

Definition: Let Ω denote the set of all possible origin–destination pairs. Thus, using standard set notation, we get the following:

$$\Omega = \{(i, j) : i, j \in \mathcal{N} \ \& \ i \neq j\}.$$

So, the total number of possible origin–destination pairs is given by $|\Omega| = n(n - 1)$. To avoid notational confusion between links in \mathcal{L} and origin–destination pairs in Ω , generic elements in Ω are often referred to as ω . Also, for each $\omega \in \Omega$, let O_ω , and let D_ω denote the origin and destination node, respectively, associated with the origin–destination pair

ω . Associated with each origin–destination pair is a demand function, as defined below.

Definition: For each $\omega \in \Omega$, there is a non-negative demand function, denoted as $r_\omega(t)$, which indicates the rate at which data are requested to be sent from node O_ω to node D_ω , at time t .

Definition: For a given network graph $G = (\mathcal{N}, \mathcal{L}, \mathcal{C})$, let \mathcal{P}_ω denote the set of all logical paths from O_ω to D_ω , defined for each $\omega \in \Omega$.

Definition: For each $\omega \in \Omega$ and each path $p \in \mathcal{P}_\omega$, let $x_p(t)$ denote the flow on path p at time t (i.e., the rate at which data is being sent along path p).

Based on the above notations and definitions, the following flow conservation and constraint equations are needed in order to define the concept of a valid routing:

$$x_p(t) \geq 0, \quad \text{for all } p \in \mathcal{P}_\omega, \omega \in \Omega, \quad (1)$$

$$\sum_{p \in \mathcal{P}_\omega} x_p(t) = r_\omega(t), \quad \text{for all } \omega \in \Omega, \quad (2)$$

$$f_{ij}(t) = \sum_{\substack{\text{all paths } p \\ \text{containing link } (i,j)}} x_p(t), \quad \text{for all } p \in \mathcal{P}_\omega, \omega \in \Omega, \quad (3)$$

$$f_{ij}(t) \leq C_{ij}, \quad \text{for all } (i,j) \in \mathcal{L}. \quad (4)$$

Equation (1) is an obvious non-negativity constraint for the flow rates on all logical paths. Equation (2) ensures that the total flow rate on all paths from O_ω to D_ω is equal to the requested demand, $r_\omega(t)$. Equation (3) states that the total flow on link (i,j) , denoted $f_{ij}(t)$, equals the sum total of all path flows $x_p(t)$ for which link (i,j) belongs. Finally, (4) states that the flow rates on each link must not exceed that link's capacity. Next the concept of a valid routing is formally defined.

Definition: For a given network graph $G = (\mathcal{N}, \mathcal{L}, \mathcal{C})$ and a given set of origin–destination demand functions, $r_\omega(t)$, $\omega \in \Omega$, a valid routing is defined as a set of path flows, $\{x_p(t) : p \in \mathcal{P}_\omega, \omega \in \Omega\}$, which satisfy (1) through (4).

Note that for a given network graph and a given set of demand functions, exactly one of the following statements is true.

- There exists at least one valid routing.
- No valid routing exists.

B. A Stochastic Model for the Demand Functions

The goal here is to provide a simple and realistic model for the demand functions,¹ $r_\omega(t)$. First, assume that time t is discretized. Second, assume that the value of the demand functions is either 0 or 1, i.e., $r_\omega(t) \in \{0, 1\}$, for all t and ω . A demand function is said to be in the active state whenever it is equal to 1, and in the inactive state whenever it is equal to 0. Now, for each ω and each t , we assign an associated Bernoulli

¹Strictly speaking, the demand functions are actually realizations of a particular stochastic process.

probability, denoted as $0 \leq q_{\omega,t} \leq 1$, which represents the probability that the demand function $r_\omega(t)$ is in the inactive state. Thus, the probability that $r_\omega(t)$ is in the active state is given by $1 - q_{\omega,t}$.

The set of all Bernoulli probabilities fully characterizes the demand functions. Thus, define the following set:

$$Q(t) = \{q_{\omega,t} : \omega \in \Omega\}$$

as a characterization of the demand functions.

It is straightforward to extend the above demand function model to where the active state can actually take on a range of strictly positive values (instead of just the value 1). To do so, however, greatly complicates the notational burden without providing any real significance to the basic results. Essentially, $r_\omega(t) = 1$ can be interpreted as a “normalized average” traffic demand of the active origin–destination pair ω , at time t . Nonuniform traffic patterns can be modeled by assigning distinct activity factors, i.e., $q_{\omega,t}$'s.

III. RECEPTIVITY

A. Defining Receptivity

For a given network graph $G = (\mathcal{N}, \mathcal{L}, \mathcal{C})$ and a given characterization of the demand functions, $Q(t)$, receptivity is defined to be the probability that a valid routing exists at time t . Later in the subsection a formal definition of receptivity is provided; first, however, a few preliminary definitions are given.

Definition: The set of active origin–destination pairs at time t , denoted as $W(t)$, is defined by

$$W(t) = \{\omega : r_\omega(t) = 1, \omega \in \Omega\}.$$

Recall that all possible sets of active origin–destination pairs can be classified in one of two ways. Namely, either 1) there exists an associated valid routing, or 2) an associated valid routing does not exist. If there exists a valid routing for a particular set of active origin–destination pairs, call the set a concurrently communicating origin–destination pair set as defined below.

Definition: Consider a network $G = (\mathcal{N}, \mathcal{L}, \mathcal{C})$ and a particular set of active origin–destination pairs $W(t)$. Define $W(t)$ as a concurrently communicating origin–destination pair set, if there exists a valid routing for the (active) demand functions $r_\omega(t)$, $\omega \in W(t)$. Also, adopt the notation $W(t) \sim W_{cc}$ to signify that $W(t)$ is a concurrently communicating origin–destination pair set.

The formal definition of receptivity is now stated.

Definition: Given a network graph $G = (\mathcal{N}, \mathcal{L}, \mathcal{C})$ and a characterization of the demand functions $Q(t)$, the receptivity of the network graph G at time t , denoted as $\rho_G(t)$, is defined by

$$\rho_G(t) = P(W(t) \sim W_{cc}).$$

In words, the receptivity of G at time t is the probability that the set of active origin–destination pairs at time t is a concurrently communicating origin–destination pair set. Obviously, the value of $\rho_G(t)$ is always between 0 and 1. So,

for example, if $\rho_G(t) = 0.95$, then the interpretation is that the network G will be able accommodate 95% of all expected concurrent communication patterns.

B. Simple Illustrative Examples

In order to appreciate the interplay between demand pattern statistics, network topology, and the associated value of receptivity, it is instructive to look at some simple examples.

Example 1: Let $G = (\mathcal{N}, \mathcal{L}, C)$ be an arbitrary link capacitated graph, and let $C_{\min} = \min_{(i,j) \in \mathcal{L}} \{C_{ij}\}$. Assume that the demand functions are characterized as follows: $q_{\omega,t} \geq 0$, for all $\omega = (i,j) \in \mathcal{L}$ and all t , and $q_{\omega,t} = 1$ for all $\omega \notin \mathcal{L}$. Next assume that $C_{\min} \geq 1$.

Because $q_{\omega,t} = 1$, for all $\omega \notin \mathcal{L}$, note that there is a zero probability of having origin–destination pairs that are not nearest neighbors request communication (i.e., be in the active state). Also, because the value of all link capacities is no less than unity, we conclude that the network is always receptive. That is, $\rho_G(t) = 1$.

Example 2: As in Example 1, let $G = (\mathcal{N}, \mathcal{L}, C)$ be an arbitrary link capacitated graph. Unlike Example 1, however, now assume that all link capacities are equal to unity, that is, let $C_{ij} = 1$, for all $(i,j) \in \mathcal{L}$. Also, the demand function characterizations (described below) are slightly different from those assumed for Example 1. Assume that $q_{\omega,t} = 0$, for all $\omega = (i,j) \in \mathcal{L}$, $q_{\omega_o,t} = q_o \neq 1$, for some $\omega_o \notin \mathcal{L}$, and $q_{\omega,t} = 1$ for all $\omega \notin \{\mathcal{L} \cup \{\omega_o\}\}$.

Putting the description of this example into words, we say that there is a zero probability that non-nearest neighbors will be in the active state, with the exception of the single origin–destination pair ω_o , which is in the active state with a probability of $1 - q_o$, where $q_o \neq 1$. Also, all active origin–destination pairs request (with probability 1) the entire capacity of a single link (which are all assumed to have unity capacity). It is easy to verify that the receptivity for this example is given by $\rho_G(t) = q_o$. This example shows how the activity of a single origin–destination pair can limit the overall receptivity of a network. It should be clear that if it is assumed that $0 < q_{\omega,t} \leq 1$, for all $\omega = (i,j) \in \mathcal{L}$ (instead of $q_{\omega,t} = 0$), then $q_o < \rho_G(t) \leq 1$.

IV. EXACT CALCULATION OF RECEPTIVITY

Calculating the exact value of a network's receptivity is not (generically) a computationally trivial task. Recall that the total number of distinct origin–destination node pairs for an n -node network is given by $|\Omega| = n(n-1)$. So, one can enumerate all possible combinations of potentially active origin–destination pair sets. For instance, there are $n(n-1)$ possible origin–destination pair sets containing only one element each. Likewise, there are the following:

$$\binom{n(n-1)}{2}$$

possible origin–destination pairs sets containing two elements each, and so on down the line. Based on this type of enumeration, the total number of potentially active origin–destination

pair sets, denoted as N_W , is given by

$$N_W = \sum_{k=0}^{n(n-1)} \binom{n(n-1)}{k} = 2^{n(n-1)}. \quad (5)$$

For convenience, label the i th possible active origin–destination pair set as W_i , where $i \in \{1, \dots, N_W\}$. Now, for a given characterization of the demand functions, i.e., $Q(t)$, one can compute the probability that the actual active origin–destination pair set at time t , denoted as $W(t)$, is equal to the i th possible origin–destination pair set, W_i . By straightforward calculation, we get

$$P(W(t) = W_i) = \prod_{\omega \in W_i} (1 - q_{\omega,t}) \prod_{\omega \in \Omega - W_i} q_{\omega,t}, \quad \text{for each } i \in \{1, \dots, N_W\}. \quad (6)$$

The above formula gives the probability that the actual set of active origin destination pairs at time t is equal to the i th possible active origin–destination pair set. For a given link capacitated graph $G = (\mathcal{N}, \mathcal{L}, C)$, we know that there either is or is not a valid routing associated with each possible active origin–destination pair set. Define an indicator function, denoted by $I_G(W_i)$, as follows:

$$I_G(W_i) = \begin{cases} 1, & \text{if } \exists \text{ a valid routing on} \\ & G \text{ associated with } W_i \\ 0, & \text{otherwise.} \end{cases}$$

So, the receptivity is computed as follows:

$$\rho_G(t) = \sum_{i=1}^{N_W} I_G(W_i) P(W(t) = W_i), \quad (7)$$

where the value of the last term is evaluated by (6).

Simply stated, (7) is the summation of the probabilities of all possible outcomes for which there exists a valid routing. So, for example, if the network G is such that there exists a valid routing for every possible W_i , except for one particular W_{i_o} , for which $P(W(t) = W_{i_o}) = 0.07$, then its associated measure of receptivity is given by $\rho_G(t) = 0.93$.

Because $N_W = 2^{n(n-1)}$, it is clear that computing the exact value of receptivity as per (7) is not practical, even for relatively small networks.

V. ESTIMATING RECEPTIVITY

A. An Upper Bound for Concurrent Communication

In this subsection, a theorem is presented that provides an upper bound for the number of origin–destination pairs for which a given network can provide concurrent communication. The bound is used later in this section as a basis for estimating a given network's measure of receptivity.

Given a graph G , let h_ω denote the shortest hop distance associated with the origin–destination pair $\omega \in \Omega$, and let ℓ_p denote the number of hops along path p , defined for all paths $p \in \mathcal{P}_\omega$, $\omega \in \Omega$. Based on the above notation, note that for each path $p \in \mathcal{P}_\omega$, the condition $\ell_p \geq h_\omega$ is obviously satisfied

for all $\omega \in \Omega$. Define x_{\min} as the minimum nonzero path flow, taken over all nonzero path flows, as follows:

$$x_{\min} = \min_{\substack{\omega \in \Omega \\ p \in \{p: x_p(t) > 0, p \in \mathcal{P}_\omega\}}} \{x_p(t)\}.$$

Theorem: For a given link capacitated graph $G = (\mathcal{N}, \mathcal{L}, \mathcal{C})$, the size of all possible concurrently communicating origin–destination pair sets, denoted by $|W_{cc}|$, is bounded above by the following condition:

$$|W_{cc}| \leq \frac{C_{\max}}{x_{\min}} \frac{|\mathcal{L}|}{h_{\text{avg}}}, \quad (8)$$

where $|\mathcal{L}|$ is the number of network links, C_{\max} is the maximum link capacity, x_{\min} is the minimum nonzero path flow (defined above), and h_{avg} is the average minimum hop distance between origin–destination pairs in W_{cc} , defined by the following:

$$h_{\text{avg}} = \frac{1}{|W_{cc}|} \sum_{\omega \in W_{cc}} h_\omega.$$

Proof: Assume that W_{cc} is an arbitrary concurrently communicating origin–destination pair set. The objective is to show that all possible choices for W_{cc} must satisfy the above inequality.

Because the origin–destination pairs $\omega \in W_{cc}$ can communicate concurrently (by assumption), then there exists (by definition) at least one associated valid routing. Now, for any such valid routing, the flow on any link (i, j) , denoted $f_{ij}(t)$, is less than the associated capacity, C_{ij} , for all $(i, j) \in \mathcal{L}$. Furthermore, because the value of $f_{ij}(t)$ is the sum of all (nonzero) path flows that contain link (i, j) , it is concluded that the maximum number of active paths sharing link (i, j) is necessarily less than or equal to $\frac{C_{ij}}{x_{\min}}$, which is bounded by $\frac{C_{\max}}{x_{\min}}$. So, the following conditions exist:

Number of Active Paths Sharing Link $(i, j) \leq$

$$\frac{C_{\max}}{x_{\min}}, \quad \text{for all } (i, j) \in \mathcal{L}. \quad (9)$$

Next note that the total number of hops taken by all active paths is given by the following:

Total Number of Hops Taken by All Active Paths =

$$\sum_{\substack{\omega \in W_{cc} \\ p \in \{p: x_p(t) > 0, p \in \mathcal{P}_\omega\}}} \ell_p, \quad (10)$$

where ℓ_p is the number of hops along path p . Now, by (9), we have the condition that each link is used by no more than $\frac{C_{\max}}{x_{\min}}$ distinct paths, so that the total number of hops taken by all active paths must be bounded by $\frac{C_{\max}}{x_{\min}} |\mathcal{L}|$. Combining this with (10), we get the following:

$$\sum_{\substack{\omega \in W_{cc} \\ p \in \{p: x_p(t) > 0, p \in \mathcal{P}_\omega\}}} \ell_p \leq \frac{C_{\max}}{x_{\min}} |\mathcal{L}|. \quad (11)$$

Now, because $|\{p : x_p(t) > 0, p \in \mathcal{P}_\omega\}| \geq 1$, for each $\omega \in W_{cc}$, and because $\ell_p \geq h_\omega$, for all $p \in \mathcal{P}_\omega$, and for each ω , we get the following:

$$\sum_{\omega \in W_{cc}} h_\omega \leq \sum_{\substack{\omega \in W_{cc} \\ p \in \{p: x_p(t) > 0, p \in \mathcal{P}_\omega\}}} \ell_p. \quad (12)$$

Combining (11) and (12) then yields the following:

$$\sum_{\omega \in W_{cc}} h_\omega \leq \frac{C_{\max}}{x_{\min}} |\mathcal{L}|. \quad (13)$$

Finally, by defining $h_{\text{avg}} = \frac{1}{|W_{cc}|} \sum_{\omega \in W_{cc}} h_\omega$, the result is proven. \square

The result of the above theorem coincides with intuition. For a fixed network topology, one would expect that if the value of C_{\max} is increased, then the potential for accommodating more concurrently communicating origin–destination pairs should increase as well. Similar statements can be made regarding the parameters x_{\min} and $|\mathcal{L}|$, i.e., increasing $|\mathcal{L}|$ and/or decreasing x_{\min} should increase the potential for more concurrent communication. Finally, the fact that the bound is inversely proportional to the average minimum hop distance of the origin–destination pairs in W_{cc} also makes intuitive sense. That is, if the set of origin–destination pairs is such that each origin–destination pair is “close” in terms of minimum hop distances, then fewer links will (generally) be used when all active origin–destination pairs are concurrently communicating. On the other hand, if the origin–destination pairs tend to be “far away” from each other, then the number of links used for communication by just a single origin–destination pair may be on the order of the diameter of the network.

In general, the value of h_{avg} is bounded above and below by the following:

$$1 \leq h_{\text{avg}} \leq d, \quad (14)$$

where d denotes the hop diameter of the network. Substituting these bounds into (8), we get the following:

$$|W_{cc}| \leq \overline{|W_{cc}|} \stackrel{\text{def}}{=} \frac{C_{\max}}{x_{\min}} \frac{|\mathcal{L}|}{1} \quad (15)$$

and

$$|W_{cc}| \leq \underline{|W_{cc}|} \stackrel{\text{def}}{=} \frac{C_{\max}}{x_{\min}} \frac{|\mathcal{L}|}{d}, \quad (16)$$

where the value of $\overline{|W_{cc}|}$ represents an “optimistic” (i.e., best case) bound, and is obtained from (8) by assuming that all active origin–destination pairs are separated by only one hop; and the value of $\underline{|W_{cc}|}$ represents a “conservative” (i.e., worst case) bound, because it is based on the assumption that all active origin–destination pairs are d hops apart. From this later bound, note that the size of concurrently communicating origin–destination pair sets is inversely proportional to the diameter of the network.

Tightness of the Upper Bound: Simulation studies presented here indicate that “on the average,” the conservative upper bound $|W_{cc}|$, of (16), is asymptotically tight (for the case of uniform traffic demand patterns). The simulations were set up as follows. First, a topological structure is chosen (linear, star, planar mesh, or random). The value of C_{max} is fixed, and the number of nodes in the network, i.e., n , is fixed. The set of active origin–destination pairs W is initialized to the empty set. Then an active origin–destination pair is chosen at random and added to the set of active origin–destination pairs. The optimal routing problem is then solved² (using the gradient projection based algorithm of [9]). If a valid routing exists (i.e., $f_{ij} < C_{ij}$, for all $(i, j) \in \mathcal{L}$), then another origin–destination pair is chosen at random and added to the set of active origin–destination pairs. This procedure is continued until a valid routing can no longer be found. At this point, the number of active origin–destination pairs is stored, and the procedure is repeated (with W reinitialized to the empty set). The average number of origin–destination pairs accommodated for each run is then computed. Denote this quantity as $|W|_{avg}$. For the linear, star, and mesh networks, 100 simulations were run for each problem size. For the random graphs, 10 simulations were done on 10 distinct, randomly generated graphs for each problem size (a total of 100 simulations for each problem size). For each topological structure and each problem size, two separate simulations were done: one with all $C_{ij} = C = 1$, and one with $C_{ij} = C = 2$.

Let $K = C/x_{min}$ in the following discussion. For the n -node linear network depicted in Fig. 1(a), note that the number of (directed) links is given by $|\mathcal{L}| = 2(n - 1)$, and that the diameter is obviously $d = n - 1$. Thus, according to (16), the worst case bound for $|W_{cc}|$ is given by the following:

$$|W_{cc}| = 2K \left(\frac{n-1}{n-1} \right) = 2K.$$

So, assuming that K is independent of n , note that $|W_{cc}|$ is a constant. Fig. 2(a) shows the simulation results for the linear network, which indicate that the simulated value of $|W|_{avg}$ is independent of n as well.

Similar numerical studies were done for the n -node star network shown in Fig. 1(b). For this network, note that the number of links is $|\mathcal{L}| = 2(n - 1)$, and that the diameter is $d = 2$. Therefore, according to the the worst case bound, we get the following:

$$|W_{cc}| = 2K \left(\frac{n-1}{2} \right) = K(n-1).$$

The results of the numerical studies for the star network are summarized by the graph of Fig. 2(b). Note that the value of $|W|_{avg}$ increases linearly with n , as expected.

To further justify the theory, simulations were done for the n -node planar mesh network (see Fig. 1(c)). In an n -node planar mesh network, $|\mathcal{L}| = K_1 n$ and $d = K_2 \sqrt{n}$; thus, according to (16), the worst case bound becomes the following:

$$|W_{cc}| = K_3 \sqrt{n},$$

²Actually, the optimal routing problem is solved only for the mesh and random topologies, because on both the linear and star networks, there is exactly one path between each pair of nodes.

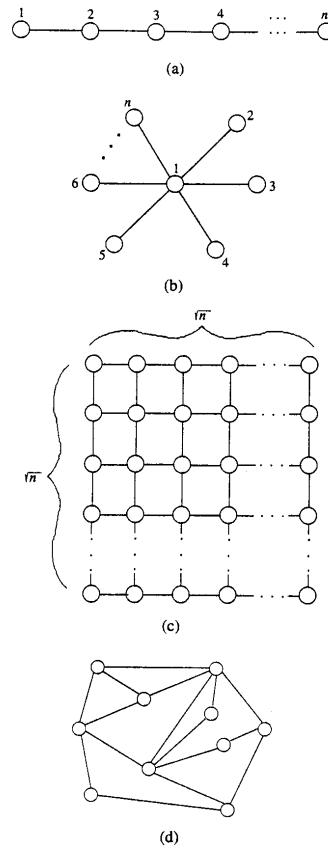


Fig. 1. (a) The n -node linear network. (b) The n -node star network. (c) The n -node planar mesh network. (Note: This network is asymmetric because wrap-around links on the boundary nodes are not included.) (d) An example of a 10-node random network.

where K_1 , K_2 , and K_3 are constants. The results of the simulations for the mesh network given in Fig. 2(c) agree with the derived bound. Namely, $|W|_{avg} = \Theta(\sqrt{n})$.

Simulations were also conducted on randomly generated topological structures (see Fig. 1(d)). The random graphs were generated by randomly selecting links from the set of $n(n-1)$ possible directed links. For an n -node graph, the probability of selecting any link (i, j) was set at $4/n$. Therefore, the expected number of links in the random graphs is $4(n-1) = \Theta(n)$, and the expected in- and out-degree at any node is approximately 4. In [13, pp. 233–236], it is proven that if the probability of selecting any link is p , then with probability 1 (as $n \rightarrow \infty$), the diameter of the graph will be equal to either d or $d + 1$, where d satisfies the following equation:

$$p^d n^{d-1} = \log(n^2/c)$$

and c is a positive constant. With $p = 4/n$, it can be shown from the above equation that $d = \Theta(\log n)$. So, for this class of random graphs, the formula for bounding concurrent

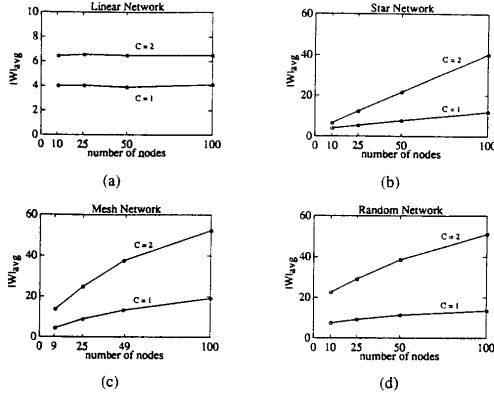


Fig. 2. (a) The results of simulation studies for the linear network. Note that the value of $|W|_{avg}$ is independent of n . (b) The results of simulation studies for the star network. Note that the value of $|W|_{avg}$ increases linearly with n . (c) The results of simulation studies for the planar mesh network. Note that the value of $|W|_{avg}$ increases with the square root of n . (d) The results of simulation studies for the random networks. Note that the value of $|W|_{avg}$ increases (approximately) with a rate $\frac{n}{\log n}$.

communication becomes the following:

$$|W_{cc}| = \frac{4(n-1)}{\Theta(\log n)} = \Theta\left(\frac{n}{\log n}\right).$$

The results of the simulations summarized in Fig. 2(d) coincide with the above expected behavior.

A. A Simple Estimate of Receptivity

The upper bound result of the previous subsection (see (8)), is used here as a basis for estimating receptivity. In order to employ the bound, however, there is the issue of how to determine the value of x_{min} . In particular, this value can be known only after a valid routing has been determined. To prevent the necessity of computing valid routings, a simplified routing formulation is assumed here. In particular, it is assumed that for each origin-destination pair ω , only one associated path is used to carry the demand $r_\omega(t)$; the remaining paths associated with ω are required to have zero flow. In the simulation studies (presented later) this "zero/one" path flow constraint is enforced by first solving the optimal routing problem by using the algorithm in [9], and then shifting all of the flow along the single path, for each origin-destination pair, having the largest fraction of flow. (It was observed that the "zero/one" routing formulation was able to accommodate about 90% of the load routed by the general "fractional flow" formulation.)

The other term to be approximated in (8) is h_{avg} . The following estimate for h_{avg} is proposed:

$$h_{avg} \approx \tilde{h}_t \stackrel{\text{def}}{=} \frac{\sum_{\omega \in \Omega} (1 - q_{\omega,t}) h_\omega}{\sum_{\omega \in \Omega} (1 - q_{\omega,t})}. \quad (17)$$

The reasoning for the above expression is that it approximates the expected value of h_{avg} taken over all W_i 's, without having to explicitly compute the value of h_{avg} for every possible W_i . In particular, if the $q_{\omega,t}$'s in (17) are equal for all ω (i.e.,

$q_{\omega,t} = q_t$, for all ω), then the approximation for h_{avg} given in (17) reduces to the original definition of h_{avg} , with $W_i = \Omega$.

Based on the approximations made thus far, the following (estimated) bound for concurrent communication results:

$$|W_{cc}| \leq \frac{C_{max} |\mathcal{L}|}{1 - \tilde{h}_t}, \quad (18)$$

where \tilde{h}_t is defined as follows:

$$\tilde{h}_t = \frac{\sum_{\omega \in \Omega} (1 - q_{\omega,t}) h_\omega}{\sum_{\omega \in \Omega} (1 - q_{\omega,t})}. \quad (19)$$

Now, if the value of $|W(t)|$ (based on some characterization of the demand functions) is, with a high probability, close to the value of the bound given for $|W_{cc}|$ in (18), then it is expected that the network would have a relatively poor measure of receptivity. On the other hand, if the value of $|W(t)|$ is, with a high probability, substantially less than the bound of (18), then it is expected that the network would have a relatively good measure of receptivity. This line of reasoning is used to justify the following estimate for receptivity:

$$\rho_G(t) \approx P\left(|W(t)| \leq \alpha \frac{C_{max} |\mathcal{L}|}{1 - \tilde{h}_t}\right), \quad (20)$$

where α is a scaling constant, $0 < \alpha \leq 1$. By choosing α sufficiently close to 1, the approximation is optimistic in that it provides a high estimate of the receptivity. On the other hand, by choosing α close to 0, the estimate is generically pessimistic in that it under estimates the actual receptivity. (Through simulation studies, it was determined that choosing $\alpha = 0.5$ typically yields a reasonable estimate.)

Unfortunately, computing the exact value of the above probability in (20) is still not trivial; it generically requires the summing of $O(2^{n(n-1)})$ of the terms given in (6). In particular, the value of $P(|W(t)| \leq k)$ is computed by using the following equation:

$$P(|W(t)| \leq k) = \sum_{\substack{i \in \{1, 2, \dots, N_W\} \\ |W_i| \leq k}} \prod_{\omega \in W_i} (1 - q_{\omega,t}) \prod_{\omega \in \Omega - W_i} q_{\omega,t}. \quad (21)$$

The next step, then, is to provide an approximation to the above probability calculation. Consider first the simple case in which the Bernoulli probabilities associated with each ω are equal. That is, assume that $q_{\omega,t} = q_t$, for all $\omega \in \Omega$. Under this assumption, it is easy to verify that the probability density function, for the size of the active origin-destination pair set, is the binomial distribution. So, we have

$$P(|W(t)| = k) = \binom{n(n-1)}{k} (1 - q_t)^k q_t^{n(n-1)-k}, \quad \text{for all } k = 0, 1, \dots, n(n-1). \quad (22)$$

Now, by the De Moivre-Laplace [10] limit theorem, (special case of the central limit theorem), if the random variable X has a binomial distribution with parameters N and $p = 1 - q$, then, for fixed $a < b$, we have

$$P(a \leq X \leq b) \rightarrow \Phi\left(\frac{b - Np}{\sqrt{Npq}}\right) - \Phi\left(\frac{a - Np}{\sqrt{Npq}}\right) \text{ as } N \rightarrow \infty, \quad (23)$$

where $\Phi(\cdot)$ is the standard cumulative normal distribution, defined by

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^z e^{-t^2/2} dt. \quad (24)$$

So, the approximation, under the assumption that $q_{\omega,t} = q_t$ for all ω and all t , becomes

$$\rho_G(t) \approx \Phi \left(\frac{\left(\alpha \frac{C_{\max} |L|}{1 h_t} \right) - n(n-1)(1-q_t)}{\sqrt{n(n-1)(1-q_t)q_t}} \right). \quad (25)$$

Now, to handle the general situation,³ approximate the actual density with a binomial distribution having parameters $n(n-1)$ and $\tilde{p}_t = 1 - \tilde{q}_t$, where \tilde{q}_t is defined by

$$1 - \tilde{q}_t = \frac{1}{n(n-1)} \sum_{\omega \in \Omega} (1 - q_{\omega,t}). \quad (26)$$

By choosing the parameter \tilde{q}_t of the approximating binomial distribution as per (26), the expected value for $|W(t)|$ (based on the approximating binomial distribution) is the same as that of the actual distribution. That is, for the actual distribution, $E[|W(t)|] = \sum_{\omega \in \Omega} (1 - q_{\omega,t})$, and for the approximating binomial distribution, $E[|W(t)|] = n(n-1)(1 - \tilde{q}_t)$. Based on all of the above approximations, we state the most general formula for estimating receptivity.

The Simple Estimate for Receptivity:

$$\rho_G(t) \approx \Phi \left(\frac{\left(\alpha \frac{C_{\max} |L|}{1 h_t} \right) - n(n-1)(1 - \tilde{q}_t)}{\sqrt{(n-1)n(1 - \tilde{q}_t)\tilde{q}_t}} \right), \quad (27.1)$$

where

$$\tilde{h}_t = \frac{\sum_{\omega \in \Omega} (1 - q_{\omega,t}) h_{\omega}}{\sum_{\omega \in \Omega} (1 - q_{\omega,t})} \quad (27.2)$$

and

$$\tilde{q}_t = 1 - \frac{1}{n(n-1)} \sum_{\omega \in \Omega} (1 - q_{\omega,t}). \quad (27.3)$$

Both \tilde{h}_t and \tilde{q}_t can be computed (serially) in $O(n^2)$ time, because $|\Omega| = n(n-1) = O(n^2)$. Also, the time required to evaluate the integral associated with $\Phi(\cdot)$ (either numerically or with a standard table) is independent of the size of the network.

C. Simulation Studies

The estimate of receptivity, given in (27), was compared with simulated values of receptivity for the four topological structures depicted in Fig. 1. In all cases, it was assumed that $q_{\omega,t} = q$, for all ω and t . So, at each point in time, the probability that any origin-destination pair is in the active state is given by $1 - q$. Figs. 3 through 6 are comparisons between estimated and simulated values of receptivity for

³By the expression "general situation," we mean a situation in which all of the values of $q_{\omega,t}$ are not equal, and thus in which the underlying density (for the size of the active origin-destination pair set) is not a true binomial distribution.

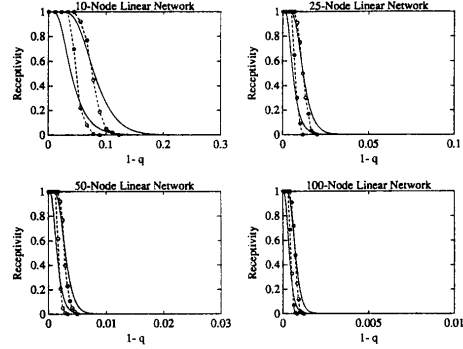


Fig. 3. Comparison of simulated and estimated values of receptivity for the linear network. Dashed lines with circles are simulated values, and solid lines are estimated values from the formula.

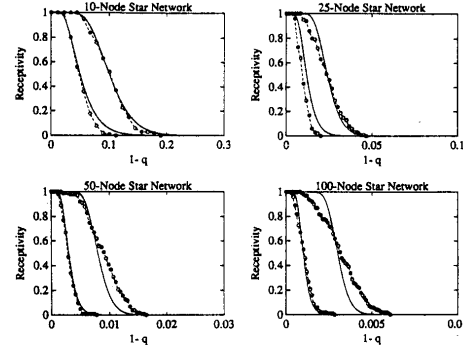


Fig. 4. Comparison of simulated and estimated values of receptivity for the star network. Dashed lines with circles are simulated values, and solid lines are estimated values from the formula.

the linear, star, planar mesh, and random topologies. The simulated values were obtained by computing (for each value of $1 - q$) the percentage of trials for which there exists a valid routing.⁴ In all cases, the upper curves correspond to using $C_{ij} = C = 2$, whereas the lower curves are associated with $C_{ij} = C = 1$. As expected, as the value of $1 - q$ is increased, the value of receptivity decreases. Also, for a fixed value of $1 - q$ and a fixed topology, the receptivity decreases as the size of the network increases, because, for all topologies considered, the total number of data links grows only as $\Theta(n)$, whereas the number of possible origin-destination pairs increases with $\Theta(n^2)$.

All computation was done on a Sun 3/60 workstation. Typical central processing unit (CPU) times required to compute simulated values of receptivity (for the 100-node networks) were on the order of several hours. The CPU times required to compute the corresponding estimated values, as per (27), were on the order of a couple of seconds. Integration of

⁴Simulated values of receptivity (instead of exact values) were computed because the number of active origin-destination pairs sets required to be tested for exact calculation, i.e., N_{IV} of (5), is extremely large, even with $n = 10$.

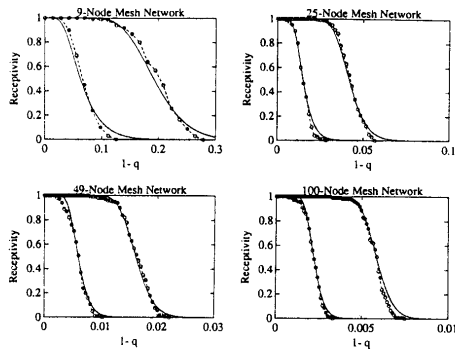


Fig. 5. Comparison of simulated and estimated values of receptivity for the planar mesh network. Dashed lines with circles are simulated values, and solid lines are estimated values from the formula.

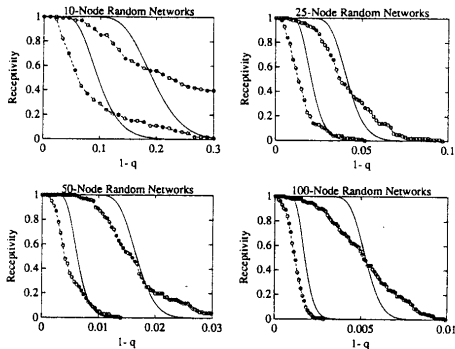


Fig. 6. Comparison of simulated and estimated values of receptivity for the random networks. Dashed lines with circles are simulated values, and solid lines are estimated values from the formula.

the standard cumulative normal distribution was accomplished with an IMSL numerical integration routine [11].

Note that for all topological structures, the estimate of receptivity at least predicts the neighborhood around which the receptivity of the network begins to decrease. Also, for the mesh structure in particular, the difference between the estimated and simulated values of receptivity is surprisingly small.

Note that both the linear and star networks each have $2(n-1)$ directed links. Because of the large diameter ($d = n-1$) of linear network as compared to the small diameter ($d = 2$) of the star network, however, the receptivity of the star network is seen to be superior. Of course, in reality, large star networks may not be practical, because of the high edge degree at the central node (which we assume can handle all required switching without conflicts).

Recall that the random graphs were generated so that the expected number of links is $4(n-1)$, which is approximately the same number of links as used by the mesh networks. For finite sized random graphs, the exact diameter is difficult to predict analytically. It is known that as $n \rightarrow \infty$, then

$d = \Theta(\log n)$ with probability 1 [13]. For the mesh networks, note that $d = 2(\sqrt{n} - 1)$. Therefore, for sufficiently large values of n , it would be expected that the receptivity of the random graphs (on the average) would be superior to that of meshes. From the simulation studies, it is difficult to draw such a conclusion, perhaps because the size of the networks was not sufficiently large. However, the simulations do suggest a trade-off in terms of the shape of the (simulated) receptivity curves. Namely, those from the random graphs tend to be flatter (more horizontal) than those associated with the meshes. So, though nominal values of receptivity of, say, $\rho = 0.5$ occur at approximately the same value of $1-q$ for both topologies, the random graphs generally provide better values of receptivity for larger values of $1-q$.

VI. CONCLUSION

One of the contributions of this paper is the definition of a performance measure for networks called receptivity. The receptivity of a network is defined as the probability that the network is able to accommodate expected concurrent communication requests. Such a metric is useful in evaluating how well a topological structure is matched to the expected demand pattern. Effective matching of the topological structure (physical and/or virtual) to the expected demand patterns will be of increasing importance for current and future high-speed WAN's, because the constant propagation delay associated with broadcasting state information implies that the effectiveness of traditional reactive control techniques will diminish as link rates increase [14].

A simple formula for receptivity is proposed, and, for the network topologies used in the simulation studies, the estimated values for receptivity (from the formula) match simulated values reasonably well. Estimated values for receptivity computed from the formula can be determined several orders of magnitude faster than can simulated values. (Exact calculation of receptivity requires an exponential-type computation time, and thus was not attempted in this paper.)

A fundamental inequality is derived for bounding the number of origin-destination pairs that a given network topology can accommodate concurrently. This bound is ultimately used as a basis for developing the simple estimate for the value of receptivity; however, the bound itself is useful in that it shows how various parameters impact the number of origin-destination pairs that a network can accommodate. The bound shows, generically, that the receptivity of a network is inversely proportional to the average hop distance between communicating origin destination pairs. So, from a topology design point of view, in order to maximize receptivity, one should strive to locate close together (in the sense of hop distance) those origin-destination pairs that are in the active state with a high probability. In future work, we plan to use this premise as a basis for characterizing effective topologies for high-speed WAN's. Two classes of network topologies that seem to be particularly well suited are the balanced hierarchically clustered topologies, defined in [6], and the sparse networks, developed in [8].

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