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Fast Solutions for a Class of Optimal Trajectory Planning Problems with Applications to Automated Spray Coating [1]

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Abstract

Optimal trajectory planning problems are often formulated as constrained variational problems. In general, solutions to variational problems are determined by appropriately discretizing the underlying objective functional and solving the resulting nonlinear programming problem(s) numerically. Determining global minima through these methods is often computationally expensive, and therefore is of limited value when optimal trajectories need to be frequently computed and/or re-computed. In this paper, a realistic class of optimal trajectory planning problems is defined for which the existence of fast numerical solution techniques are demonstrated. To illustrate the practicality of this class of trajectory planning problems and the proposed solution techniques, three optimal trajectory planning problems for spray coating applications are formulated and solved.

1. Introduction

This paper addresses solution techniques for a class of trajectory planning problems that arise in manufacturing applications. The discussion is motivated by a particular problem in spray coating applications, where the objective is to determine the optimal time profile for a spray coating applicator that is constrained to traverse a specified spatial path.

In large-scale production lines, spray coating applicators are attached to robotic manipulators that move the applicator around the surface to be coated. Experienced operators of such systems can often provide good choices for the spatial path of the robot's end-effector. A less intuitive issue (than selecting effective spatial paths) is to decide how to traverse a given spatial path temporally (i.e., with respect to time). In general, the accumulated film thickness of a target area is proportional to the amount of time spent spraying the area. Therefore, moving the applicator more slowly over certain regions may be called for if the spatial path is such that there is very little accumulation contributed to the area by other positions on the path.

Studies in [1, 3] discuss general methods that are applicable for automatically determining both the spatial and temporal components of the applicator's trajectory using nonlinear programming methods. In the present paper, the focus is on determining the optimal time profile of an applicator that is constrained to traverse a specified spatial path. Although the "time and space" formulations of the past (i.e., [1, 3]) can be applied to the restricted problem of finding the optimal time profile for traversing a specified spatial path, they still generally result in nonlinear (and nonconvex) programming problems. In contrast, an alternative formulation is proposed here for the restricted problem that results in either linear or quadratic programming problems, depending on the specific objective function assumed.

It is assumed that the positions along a spatial path are characterized by a continuous vector function $\mathbf{p}(\lambda)$, where the elements of $\mathbf{p}(\lambda)$ define the coordinates of the applicator as a function of the scalar parameter λ. It is further assumed that the spatial path is parameterized by arc length, which means that a unit change in the parameterizing variable λ results in a unit change in curve length along the path [2]. For this type of parameterization, $\lambda \in [0, L]$, where L is the total length of the path. To model the motion of the applicator along a parameterized path during a time interval $[0, T]$, the scalar quantity λ is replaced by a scalar function of time $\psi(t)$, where $\psi : [0, T] \rightarrow [0, L]$. Therefore, the position of the applicator at a given instant of time t is specified by $\mathbf{p}(\psi(t))$. The function $\psi(t)$ is referred to as the time profile of the applicator.

The cost functional and any constraint functionals for spray coating are typically associated with one or more process performance metrics such as painting time, variation in film thickness, average film thickness, expended paint, and transfer efficiency. When the spatial path is specified, the problem is to determine the function $\psi(t)$ to satisfy the performance constraints and optimize a specified performance index associated with the cost functional. The following optimization problems are considered in this pa-

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per: (1) minimize painting time subject to achieving a specified average thickness; (2) minimize variation in film thickness subject to achieving a specified average thickness, and (3) minimize variation in film thickness subject to achieving a specified average thickness and an upper bound on painting time. Although the paper addresses methods for these specific problems, the framework developed can also be applied to other performance objectives.

2. Basic assumptions and definitions

The surface to be coated is defined by a set of points $\mathcal{S} \subset \mathcal{R}^3$. The set of points along the parameterized spatial path $\mathbf{p}(\lambda)$ (which defines the positions at which the applicator is constrained to be located) is defined by $\mathcal{A}_\mathbf{p} = \{\mathbf{a} : \mathbf{a} = \mathbf{p}(\lambda), \lambda \in [0, L]\}$. It is assumed that the orientation of the applicator is specified for each point in this set. A typical specification in spray coating is to orient the applicator normal to the surface that is to be coated. A mapping, $f : \mathcal{S} \times \mathcal{A}_\mathbf{p} \to \mathcal{R}^+$ is assumed, which defines the rate of film accumulation at each surface point $\mathbf{s} \in \mathcal{S}$ for each possible location of the applicator $\mathbf{a} \in \mathcal{A}_\mathbf{p}$. Therefore, $f(\mathbf{s}, \mathbf{p}(\psi(t)))$ represents the rate of film accumulation for each surface point $\mathbf{s} \in \mathcal{S}$ at time t, where the applicator traverses the parameterized spatial path according to $\mathbf{p}(\psi(t))$, and $\psi : [0, T] \to [0, L]$.

The film thickness (for each surface point $\mathbf{s}$) accumulated over the time period $[0, T]$ is denoted by $F(\mathbf{s}, \mathbf{p}(\cdot), \psi(\cdot), T)$ and is obtained by integrating the assumed film accumulation rate function over the time period $[0, T]$:

$$F(\mathbf{s}, \mathbf{p}(\cdot), \psi(\cdot), T) = \int_0^T f(\mathbf{s}, \mathbf{p}(\psi(t)))\, dt. \quad (1)$$

The basic assumption made here is that, for a given a set of distinct positions of the applicator along a specified path, the corresponding film accumulation rate at the surface points (characterized by the mapping f) is known. This mapping can be based on theoretical models and/or be derived from empirical data collected through off-line experimentation.

Two important measures of quality that are used in the optimization problems considered in this paper are: (1) the average film thickness and (2) the variation in film thickness over the surface. These quantities, which characterize the deposition of paint over a surface, depend on the film thickness function given in Equation 1.

The average film thickness, $G(\mathbf{p}(\cdot), \psi(\cdot), T)$, is obtained by integrating the expression for film thickness over the entire surface and dividing by the area of the surface:

$$G(\mathbf{p}(\cdot), \psi(\cdot), T) = \frac{1}{A_S} \int_S F(\mathbf{s}, \mathbf{p}(\cdot), \psi(\cdot), T)\, d\mathbf{s}, \quad (2)$$

where,

$$A_S = \int_S d\mathbf{s}. \quad (3)$$

The variation in film thickness, $\mathcal{V}(\mathbf{p}(\cdot), \psi(\cdot), T)$, is obtained by integrating the squared difference between the actual and the average thickness over the entire surface and dividing the area of the surface:

$$\begin{aligned}\mathcal{V}(\mathbf{p}(\cdot), \psi(\cdot), T) &= \frac{1}{A_S} \int_S [F(\mathbf{s}, \mathbf{p}(\cdot), \psi(\cdot), T) \\ &\quad - G(\mathbf{p}(\cdot), \psi(\cdot), T)]^2\, d\mathbf{s}. \quad (4)\end{aligned}$$

The expression for film thickness (Equation 1) appears in both of these performance indicators (Equations 2 and 4). In optimization problems where the objective and/or constraints are based on expressions such as these, which depend on the film thickness function, determining an appropriate representation for the film thickness function in terms of $\psi(\cdot)$ is important. This issue is studied in the next section.

3. Approximate expressions for the film thickness function

One difficulty in solving optimization problems involving the film thickness function is due to the fact that, in many cases, analytical expressions for the film thickness function (in terms of $\psi(t)$) are either not possible to compute or difficult to determine. Computing the film thickness function involves the integration of the film accumulation rate function, which is typically a nonlinear function of $\psi(\cdot)$.

By approximating the film thickness function using an appropriate discretization technique, the given variational problem in $\psi(\cdot)$ reduces to a finite dimensional optimization problem. Standard discretization approaches, which involve discretizing time, result in an expression that is nonlinear with respect to the associated discrete variables (e.g., see [1]). An alternative discretization is used in this paper, yielding a relation for the film thickness linear with respect to it's discrete set of variables. The linearity of the alternative expression for film thickness enables the corresponding expressions for average thickness and variation in film thickness to be expressed as linear and quadratic functions, respectively.

The proposed approximation

The proposed approximation for the film thickness function is based on utilizing an alternative discretization of the time profile function. In this alternative approach, a finite number of evenly spaced points along the spatial path are considered and the amount of time spent at each of these spatial points are used as variables. This is in contrast to the discretization used in [1], where a finite number of evenly-spaced time instances are considered and the spatial positions for each of these time instants are used as variables. The basis for the approximation (which is well defined only when $\psi(\cdot)$ is monotone) is provided by the following theorem.

Theorem 1: Given a spatial path parameterization $\mathbf{p}(\cdot)$, for every continuously differentiable time profile function $\psi(\cdot)$, $\psi : [0,T] \to [0,L]$, there exists a monotone time profile function $\bar{\psi}(t)$ such that

$$F(\mathbf{s},\mathbf{p}(\cdot),\psi(\cdot),T) = F\left(\mathbf{s},\mathbf{p}(\cdot),\bar{\psi}(\cdot),T\right), \quad \text{for all } \mathbf{s} \in \mathcal{S}. \tag{5}$$

Proof: Refer to [7]. ∎

Theorem 1 provides the justification for the proposed alternative formulation. As every non-monotone time profile has a corresponding monotone time profile that generates the same film thickness function, the search space for any associated optimization problem can be reduced to the set of monotone time profile functions. As a result, the following expression can be used to represent all possible film thickness functions using monotone time profiles, as opposed to Equation 1, which represents film thickness functions for arbitrary time profiles.

$$F\left(\mathbf{s},\mathbf{p}(\cdot),\bar{\psi}(\cdot),T\right) = \int_0^L f(\mathbf{s},\mathbf{p}(\lambda))\left(\frac{d\bar{\psi}^{-1}(\lambda)}{d\lambda}\right)d\lambda. \tag{6}$$

Instead of directly searching for the function $\psi(t)$ as required by Equation 1, the formulation of Equation 6 is based on determining a monotone function $\bar{\psi}^{-1}(\lambda)$. This is done by searching for an appropriate function $\frac{d\bar{\psi}^{-1}(\lambda)}{d\lambda}$ that is positive (which ensures that $\bar{\psi}^{-1}(\lambda)$ and $\bar{\psi}(t)$ are monotone). Denoting $\frac{d\bar{\psi}^{-1}(\lambda)}{d\lambda}$ by $\nu(\lambda)$, Equation 6 can be rewritten as

$$F\left(\mathbf{s},\mathbf{p}(\cdot),\bar{\psi}(\cdot),T\right) = \int_0^L f(\mathbf{s},\mathbf{p}(\lambda))\nu(\lambda)d\lambda. \tag{7}$$

To approximate the integral of Equation 7, the spatial interval $[0,L]$ is divided into N sub-intervals, where each sub-interval is of width $\delta = L/N$. Assuming $\mathbf{p}_i = \mathbf{p}((i-1)\delta + \delta/2)$ to be known (the spatial path $\mathbf{p}(\cdot)$ is assumed to be given), the expression for the film thickness function in Equation 6 can be approximated using this discretized representation as

$$F(\mathbf{s},\mathbf{p}(\cdot),\psi(\cdot),T) \approx \sum_{i=1}^{N} f(\mathbf{s},\mathbf{p}_i)\,\delta\nu_i. \tag{8}$$

The quantity $\delta\nu_i$ represents the time spent by the applicator in the i-th spatial sub-interval. This is because the values of ν_i represent the reciprocal of the applicator's speed over the i-th spatial sub-interval ($\nu(\lambda) = \frac{d\bar{\psi}^{-1}(\lambda)}{d\lambda} = \frac{1}{d\bar{\psi}(t)/dt}$) and δ is the width of each spatial sub-interval. Denoting the quantity $\delta\nu_i$ by τ_i, the expression for the film thickness function is given by

$$\begin{aligned}\tilde{F}_a(\mathbf{s},\mathbf{p}(\cdot),\Gamma,T) &= \sum_{i=1}^{N} f(\mathbf{s},\mathbf{p}_i)\,\tau_i \\ &\approx F(\mathbf{s},\mathbf{p}(\cdot),\psi(\cdot),T),\end{aligned} \tag{9}$$

where $\Gamma = [\tau_1,\tau_2,\cdots,\tau_N]'$ represents the vector of discrete variables for the proposed approximation.

Equation 8 illustrates the simplification that results from the proposed formulation. The new expression (approximation) for the film thickness function is a linear function of the unknown variables. A standard approximation as outlined in [1] results in a nonlinear equation in the unknown variables. Using the expression for the film thickness function in Equation 8, the associated average thickness function (Equation 2) is approximated as

$$\begin{aligned}\tilde{G}_a(\mathbf{p}(\cdot),\Gamma,T) &= \sum_{i=1}^{N}\tau_i\frac{1}{A_S}\int_S f(\mathbf{s},\mathbf{p}_i)\,d\mathbf{s} \\ &\approx G(\mathbf{p}(\cdot),\psi(\cdot),T).\end{aligned} \tag{10}$$

By defining $g_i = \frac{1}{A_S}\int_S f(\mathbf{s},\mathbf{p}_i)d\mathbf{s}$, and denoting the vector of all g_i's by $\mathbf{g} = [g_1,g_2,\cdots,g_N]'$, the approximate average thickness function can be expressed as

$$\tilde{G}_a(\mathbf{p}(\cdot),\Gamma,T) = \sum_{i=1}^{N} g_i\tau_i = \mathbf{g}'\Gamma, \tag{11}$$

where the g_i's are constant coefficients.

Similarly, the variation in film thickness, $\mathcal{V}(\mathbf{p}(\cdot),\psi(\cdot)$ as expressed in Equation 4, can be approximated as

$$\begin{aligned}\tilde{\mathcal{V}}_a(\mathbf{p}(\cdot),\Gamma,T) &= \frac{1}{A_S}\int_S\left(\sum_{i=1}^{N} f(\mathbf{s},\mathbf{p}_i)\,\tau_i\right)^2 d\mathbf{s} \\ &\quad -\left(\sum_{i=1}^{N} g_i\tau_i\right)^2 \\ &\approx \mathcal{V}(\mathbf{p}(\cdot),\psi(\cdot),T).\end{aligned} \tag{12}$$

4. Three optimization problems

The three optimization problems discussed in Section 1 are formulated based on the proposed approximation for film thickness, average thickness, and variation in film thickness developed in the previous section. The optimization problems considered are: (1) minimize painting time subject to achieving a specified average thickness; (2) minimize variation in film thickness subject to achieving a specified average thickness, and (3) minimize variation in film thickness subject to achieving a specified average thickness and an upper bound on painting time. Because achieving a specified average thickness is a common constraint, the three problems are referred to as minimum painting time, minimum variation, and time constrained minimum variation problems, respectively.

The minimum painting time problem

Recall the expression for the average thickness in Equation 11. As the sum of the associated unknown variables (the τ_i's) represents the total painting time T, the problem in this framework is formulated as

$$\min_{[\tau_1,\tau_2,\cdots,\tau_N]} \left\{ \sum_{i=1}^{N} \tau_i \right\} \tag{13}$$

$$\text{subject to} \quad \tilde{G}_a\left(\mathbf{p}(\cdot), \Gamma, T\right) = H \tag{14}$$

$$\text{and} \quad \tau_i \geq 0, \quad \forall i, \tag{15}$$

where H is a desired average thickness value. The minimum painting time problem thus becomes a standard linear programming problem:

$$\min_{\Gamma}\{\mathbf{1}'\Gamma\} \tag{16}$$

$$\text{subject to} \quad \mathbf{g}'\Gamma = H \tag{17}$$

$$\text{and} \quad \Gamma \geq 0, \tag{18}$$

where $\mathbf{1} = [1, 1, \cdots, 1]'$.

Suppose the maximum of the elements of the vector $\mathbf{g}$ is at the q-th index, the solution is written as

$$\tau_i = \begin{cases} 0 & \text{if } i \neq q \\ \frac{H}{g_q} & \text{if } i = q. \end{cases} \tag{19}$$

The physical implication of this solution is to have the applicator spray the surface from one point, until the specified average thickness H is reached. Though this solution is unrealistic in terms of an actual implementation, the absolute minimum time necessary to achieve a specified average thickness is determined. This provides the lowest possible time bound for the time constrained minimum variation problem.

The minimum variation problem

Given a parameterized spatial path $\mathbf{p}(\cdot)$, and an associated film accumulation rate function characterized by the mapping f, the objective of the minimum variation problem is to determine the time profile that causes the variation in film thickness to be minimized, subject to achieving a specified average thickness H over the given surface. Using the approximations for the average thickness and variation in film thickness (Equations 11 and 12), the minimum variation problem can be expressed as a quadratic program in Γ. The problem is posed as

$$\min_{[\tau_1,\tau_2,\cdots,\tau_N]} \{\tilde{V}_a\left(\mathbf{p}(\cdot), \Gamma, T\right)\} \tag{20}$$

$$\text{subject to} \quad \tilde{G}_a\left(\mathbf{p}(\cdot), \Gamma, T\right) = H \tag{21}$$

$$\text{and} \quad \tau_i \geq 0, \quad \forall i, \tag{22}$$

where H is a desired average thickness value. For convenience of notation, define a matrix P, such that the $[i, j]$-th element of P is given by

$$P_{[i,j]} = \frac{1}{A_S} \int_S f\left(\mathbf{s}, \mathbf{p}_i\right) f\left(\mathbf{s}, \mathbf{p}_j\right) ds. \tag{23}$$

Using this notation to express the objective function, the minimum variation problem is expressed as

$$\min_{\Gamma}\{\Gamma' P \Gamma - \Gamma' \mathbf{g}\mathbf{g}' \Gamma\}, \tag{24}$$

$$\text{subject to} \quad \mathbf{g}'\Gamma = H \tag{25}$$

$$\text{and} \quad \Gamma \geq 0. \tag{26}$$

The two constraints that are imposed on the solutions are the average thickness equality constraint and the constraint that the time values are positive. These can be written in the form of a single vector inequality given as

$$\underbrace{\begin{bmatrix} I_N \\ \hline -\mathbf{g}' \\ \hline \mathbf{g}' \end{bmatrix}}_{K} \underbrace{\begin{bmatrix} \tau_1 \\ \tau_2 \\ \vdots \\ \tau_N \end{bmatrix}}_{\Gamma} \geq \underbrace{\begin{bmatrix} 0 \\ \hline -H \\ \hline H \end{bmatrix}}_{e}, \tag{27}$$

where I_N denotes the identity matrix. The constrained problem is written as a quadratic program in the following form:

$$\min_{\Gamma}\left\{\Gamma(P - \mathbf{g}\mathbf{g}')\Gamma'\right\} \tag{28}$$

$$\text{subject to} \quad K\Gamma \geq e. \tag{29}$$

This can be solved by standard quadratic programming routines [6].

The time constrained minimum variation problem

The time constrained minimum variation problem involves the addition of an upper bound constraint on painting time. The constraint on painting time has to be introduced explicitly in the quadratic program described in the previous section. Nevertheless, as an upper bound on painting time is also a linear constraint, the quadratic structure of the program is not destroyed. The constraint can be appended as an extra row to the K and e matrices.

5. Numerical studies

The optimization problems discussed in Section 4 are solved based on the film accumulation rate model presented in [5]. A flat panel of dimensions $5\frac{1}{3} \times 5\frac{1}{3}$ is used as the surface on which a specified average thickness is to be achieved. The applicator is assumed to traverse a path that lies on a plane above, and parallel to, the panel, at a unit distance. The analytical parameterization of the spatial path is given in [7]. The spatial path is shown in Figure 1. A value of $N = 74$ was used in all simulations.

Comparative optimization Studies

The studies described in Table 1 compare the standard nonlinear programming approach (NP) used in [1] to the proposed approach for determining the time profile. The minimum time problem is solved by a linear program (LP) and the minimum variation problem is solved by a quadratic program (QP). The significant difference in the CPU time and quality of results (Table 1) demonstrates the superiority of the proposed approach. For the NP solutions, an IMSL routine BCONF [4] was used with a constant speed initial guess (i.e., $\psi(t) = (t/T)L$), which was the best initial guess for the nonlinear programming solution. The CPU time is given in seconds. All simulations were done on a Sun SPARCstation 5.

Simulation	Min. Time		Min. Variation	
	NP	LP	NP	QP
Optimal Index	15.94	15.94	0.2899	0.1822
CPU Time	2959.78	3.65	23679.9	63.73

Table 1: Comparison of solutions and CPU times.

Process optimization studies with the alternate Formulation

To further illustrate the utility of the proposed formulation, two studies were conducted. The first study compares the performance of the minimum time and minimum variation solutions, in terms of variation in film thickness and total painting time. (For the minimum variation problem, no constraints were imposed on painting time.) The results of this study are summarized in Table 2. The performance of a constant speed trajectory is also tabulated for comparison[2]. From Table 2, it is seen that the minimum time solution has the highest variation in film thickness of all the solutions. The constant speed solution has a better variation in film thickness but requires more painting time. The trend continues for the minimum variation solution, where the variation is the least but the painting time is highest. Note that no constraints were placed on the painting time for this particular solution. The average thickness is constrained to be unity for all cases.

Type of Solution	Variation in Film Thickness	Painting Time
Minimum time	12.23	15.94
Constant speed	0.1901	28.19
Minimum variation	0.1822	39.98

Table 2: A comparison of minimum time, constant speed, and minimum variation solutions (minimum time and minimum variation solutions were generated by the proposed formulation).

The second study, results of which are presented in Table 3, involves the comparison of minimum variation solutions, with constraints imposed on total time. Recall from Table 2 that the total painting time for the minimum variation solution is more than that of the constant speed solution by about 40%. In industrial production lines, this difference may add up to a significant amount of "excess" finishing time. The motivation for imposing time constraints is to study the tradeoff between painting time and quality, as measured by the variation in film thickness. Two cases are presented in the study. First, the time taken for the constant speed case is used as an upper bound for painting time. Second, the time bound is lowered, so that the constant speed case is not a feasible solution. From Table 3, note that there exists a solution $\psi(\cdot)$ that results in a better variation in film thickness than the constant speed solution, with less painting time.

Type of Solution	Variation in Film Thickness	Painting Time
Painting time ≤ 28.19		
Constant speed	0.1901	28.19
Minimum variation	0.1843	28.19
Painting time ≤ 26		
Constant speed	—	—
Minimum variation	0.1852	26

Table 3: Comparison of solutions generated by the proposed formulation with constraints imposed on painting time.

[2]The constant speed solution corresponds to the case where equal units of time are spent along each segment of the spatial path. Thus, if the desired average thickness is H units, the time spent at each segment is given by

$$\tau_i = \frac{H}{1'g} \quad \forall i.$$

In this case study, it was observed that, for the unconstrained case, the applicator spends a significant amount of time at points along the curved portions of the trajectory where the rate of film accumulation on the surface is low [7]. This is done in order to reduce the variation in film thickness near the edges of the surface (the curved portions of the spatial path are not directly above the plate). When a constraint is placed on painting time, the applicator is caused to spend less time at the curved portions of the path shown in Figure 1. When the applicator is moving along these portions, the resulting deposition rate on a given point on the surface is very small. Therefore, the best method to achieve a better variation in film thickness with a painting time constraint is to spend less time along the curved portions. This causes a reduction in painting time, while achieving a good variation in the film thickness. Thus, for the constrained case (total time not more than 26 units), the variation in film thickness is better than the constant speed time profile solution, which is now impossible because of the strict time constraint. (For more details on solution behavior, see [7].)

6. Conclusions

A class of optimal trajectory planning problems has been discussed with applications to automated spray coating. Conventional formulations for these applications generally yield nonlinear programming problems that are computationally expensive. The formulation developed in this paper is shown to yield linear or quadratic programming problems. The solution procedures are evaluated through simulation studies, and comparisons are made with earlier work from the literature. In the simulation studies, two separate optimization subroutines developed by IMSL Corporation (one specifically for quadratic programming problems, the other for general nonlinear programming problems) are used. It is shown that the quadratic programming problem associated with the proposed approach can be solved up to three orders of magnitude faster than the general nonlinear program required for the standard approach.

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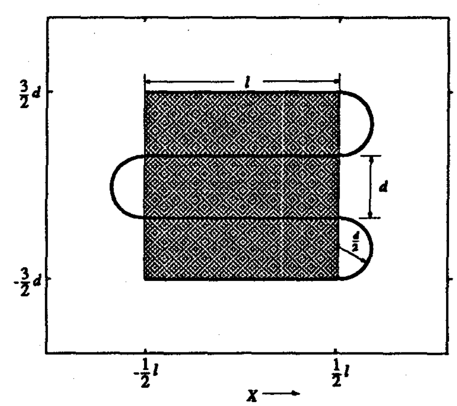

Figure 1: The spatial path chosen for the simulation studies conducted in this paper. The path is used to traverse over a square plate, indicated by the shaded area.