Optimal Trajectory Planning for Spray Coating*

John K. Antonio
School of Electrical Engineering
Purdue University
1285 Electrical Engineering Building
West Lafayette, IN 47907-1285

Abstract—The problem of how to optimally traverse a spray applicator around a surface to be coated is formulated as a type of optimization problem known as a constrained variational problem. An optimal trajectory for a spray applicator is defined to be one that results in minimal variation in accumulated film thickness on the surface. For each surface point and for each feasible position and orientation of the applicator, a value for the instantaneous rate of film accumulation is assumed to be known. Empirical data and/or estimates for these values can be readily incorporated in the formulation. By making realistic approximations, the proposed constrained variational problem is transformed into a finite dimensional constrained optimization problem. Numerical studies are included that illustrate the utility of the problem formulation and the effectiveness of applying standard nonlinear programming techniques for determining solutions.

I. Introduction

A. Background

The perceived quality of products such as automobiles, appliances, and furniture can be strongly influenced by the quality of their painted surfaces. Spray applicators are commonly used in industry to apply paint to the surfaces of manufactured products. The task of consistently achieving high quality finishes from spray applicator systems is complicated by the sensitivity of the coating process relative to environmental conditions (e.g., ambient temperature, barometric pressure, and relative humidity) and parameters associated with the spray system itself (e.g., position and orientation of the applicator, paint flow rate, atomizing air pressure, and paint viscosity).

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The process of spray coating involves first the atomization and then the spraying of a coating material (e.g., paint) toward a surface to be coated. Paints typically contain some type of solvent. As the solvent evaporates, liquid paint becomes more viscous; it eventually becomes solid when all solvent has evaporated. As atomized droplets of paint are transported through the air from the applicator to the surface, a relatively large fraction of solvent evaporates from the droplets, because the ratio of surface area to volume is relatively high for small droplets. Therefore, by the time droplets strike the surface, the viscosities of the droplets are substantially larger than they were immediately after atomization. This increase in viscosity helps to prevent the paint from running and/or sagging on the surface [1].

The hue of a surface that is coated with a colored paint depends (to a degree) on the film thickness of the paint. The film should be sufficiently thick so as to "hide" the influence of the color associated with the underlying primer coating (or the color of the surface itself if no primer coating is present). Thus, one way to produce a uniform hue across a surface is to accumulate a sufficient amount of film thickness at each surface point, i.e., enough thickness at each surface point to hide the primer. However, this approach can result in wasted paint if film thickness is not kept uniform across the surface. Also, those portions of the film that are too thick have the undesirable tendency to crack in use [13].

B. Automated Spray Painting

To specify a trajectory for a robotic manipulator, it is common practice for an operator to literally "teach" the robot a path by grasping the end-effector and manually moving the end-effector around the part to be painted while the robot's control computer records position and orientation information [13]. Having stored the path information, the robot can then repeatedly traverse the "learned" path using a speed profile spec-

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ified by the operator.

In this paper, the question of how to optimally traverse a spray applicator around a surface to be coated is formulated as a type of optimization problem known as a constrained variational problem. An optimal trajectory is defined here as one that results in minimal variation in film thickness on the surface. While other factors besides uniformity of film thickness also contribute to the overall quality of the finish, minimizing variation in film thickness is known to be a desirable property for many applications, see for example [3, 13].

The trajectory for an applicator is defined by a six-dimensional vector function that specifies the position and orientation of the applicator at each instant of time. The surface to be coated is assumed to be represented with a function. For each surface point and for each feasible position and orientation of the applicator, a value for the instantaneous rate of film accumulation is assumed to be known. Empirical data and/or estimates for these values can be readily incorporated in the formulation. For more discussion related to the importance and practicality of using empirical data for film accumulation rates, refer to [2, 14].

C. *Organization of the Paper*

The remainder of the paper is organized in the following manner. In Section II, a model for the spray coating process is described and some associated notations are introduced. The mathematical formulation of the optimal trajectory planning problem is developed in Section III. In Section IV, solution techniques for the formulated optimization problem are developed for two classes of the assumed feasible set of applicator trajectories. Section V includes simulation studies that demonstrate the effectiveness of standard numerical techniques in providing solutions that achieve the desired objective.

II. Modeling and Notation

A. *The Surface Model*

The object to be coated is assumed to be stationary and its location and surface geometry in three-dimensional euclidean space are described relative to a fixed reference frame XYZ. The surface is assumed to be representable by a function $z = h(x, y)$, where the mapping $h : \mathcal{D} \rightarrow \Re$ and its domain $\mathcal{D} \subset \Re^2$ are specified. Applying standard set notation, the surface associated with the function h is defined as

$$\mathcal{S}_h = \{(x, y, z) : z = h(x, y), \text{ for all } (x, y) \in \mathcal{D}\}. \quad (1)$$

The assumption of having a functional representation for the surface of the object, i.e., $h(x, y)$, is not unrealistic for many applications. For instance, in the automobile manufacturing industry, CAD models for surfaces are often a result of the design phase, and therefore a mathematical representation for the surface may already be known. There are several popular methods for representing geometric surfaces including the use of Coons/Ferguson patches, Bezier surfaces, and B-splines. For a more detailed description of these and other geometric modeling techniques, refer to [5, 12]. As it is not the intended thrust of the present paper to discuss how to convert various types of CAD models into the form $z = h(x, y)$, the existence of a function $h(x, y)$ will henceforth be assumed with the realization that in practice some extra effort may be required for converting any particular CAD description into this form.

B. *The Applicator Trajectory*

The spatial position and orientation of the applicator with respect to the fixed reference frame is defined by six values: three coordinate values for its position and three angular values for its orientation. These six values are defined at time t by a vector function:

$$a(t) = [a_x(t) \;\; a_y(t) \;\; a_z(t) \;\; a_\psi(t) \;\; a_\theta(t) \;\; a_\phi(t)]'. \quad (2)$$

The values $a_x(t)$, $a_y(t)$, and $a_z(t)$ represent the applicator's position at time t with respect to the fixed euclidean reference frame XYZ. The values $a_\psi(t)$, $a_\theta(t)$, and $a_\phi(t)$ describe the applicator's angular rotation with respect to the X, Y, and Z axes, respectively. This particular system of eulerian angles of rotation about the axes of the fixed reference frame is commonly referred to as the "roll, pitch, and yaw" system. It is a simple matter to define a rotation matrix based on these angular values that can be used to transform the fixed reference frame to a rotated reference frame attached to the applicator [7].

C. *The Rate of Film Accumulation*

The rate at which film accumulates at each surface point is assumed to be dependent only on the geometry of the surface and the position and orientation of the applicator relative to the surface. While the rate of film accumulation at each surface point is also a function of other parameters such as the flow rate of the coating material, the atomizing pressure, electrostatic voltage (if applicable), viscosity of the coating material, and solvent concentration, for the study here, these other parameter values will assumed to be fixed.

Let $\dot{f}_{\mathcal{S}_h}(a(t), x, y, t)$ denote the rate of film accumulation at time t at the point $(x, y, h(x, y))$ on the surface $\mathcal{S}_h$, with the applicator trajectory defined by $\boldsymbol{a}(t)$. As the notation suggests, for a given surface $\mathcal{S}_h$, the rate of film accumulation at a point on the surface depends on the "x, y" coordinates of the surface point and on time "t," which captures the position and orientation information for the applicator through the trajectory vector $\boldsymbol{a}(t)$. The values of $\dot{f}_{\mathcal{S}_h}(\boldsymbol{a}(t), x, y, t)$ may be derived from tabulated data (based on experimental measurements) and not necessarily expressed as an analytic function.

III. The Optimal Trajectory Planning Problem (OTPP)

A. *The Objective of the OTPP*

The objective of the OTPP is to determine a trajectory that results in minimal variation in accumulated film thickness on the surface. The specific objective used here is the mean squared error between actual film thickness and average film thickness across the surface.

For a trajectory $\boldsymbol{a}(t)$ defined over a time interval $[0, T]$, the film thickness accumulated during the time interval $[0, T]$ at each point $(x, y, h(x, y))$ on the surface $\mathcal{S}_h$ is given by

$$f_{\mathcal{S}_h}(\boldsymbol{a}(t), x, y) = \int_0^T \dot{f}_{\mathcal{S}_h}(\boldsymbol{a}(t), x, y, t)\, dt. \quad (3)$$

Due to the integration over time, the accumulated film thickness $f_{\mathcal{S}_h}(\boldsymbol{a}(t), x, y)$ does not depend explicitly on t; however, it does depend on the vector function $\boldsymbol{a}(t)$. The total volume of paint deposited onto the surface is given by

$$F_{\mathcal{S}_h}(\boldsymbol{a}(t)) = \iint_{\mathcal{D}} f_{\mathcal{S}_h}(\boldsymbol{a}(t), x, y)\, dx\, dy. \quad (4)$$

If the values for $\dot{f}_{\mathcal{S}_h}(\boldsymbol{a}(t), x, y, t)$ are derived from a collection experimentally measured data values and/or not expressed analytically, then the integration required in Eqs. (3) and (4) can be approximated numerically by using standard numerical integration techniques.

The area of the surface is given by [8]:

$$A_{\mathcal{S}_h} = \iint_{\mathcal{D}} \sqrt{1 + \left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial y}\right)^2}\, dx\, dy, \quad (5)$$

where it is implicitly assumed that the functional description of the surface, i.e., $h(x, y)$, can be partioned into a finite number of smooth sub-surfaces. The average film thickness over the surface, denoted as $f^{\text{avg}}_{\mathcal{S}_h}(\boldsymbol{a}(t))$, is defined as the total volume of paint deposited onto the surface divided by the area of the surface:

$$f^{\text{avg}}_{\mathcal{S}_h}(\boldsymbol{a}(t)) = \left(\frac{1}{A_{\mathcal{S}_h}}\right) F_{\mathcal{S}_h}(\boldsymbol{a}(t)). \quad (6)$$

The variation in film thickness for the surface, denoted by $V_{\mathcal{S}_h}(\boldsymbol{a}(t))$, is defined as the mean squared error between film thickness at each point and average film thickness:

$$V_{\mathcal{S}_h}(\boldsymbol{a}(t)) = \frac{1}{A_{\mathcal{S}_h}} \iint_{\mathcal{D}} \left(f_{\mathcal{S}_h}(\boldsymbol{a}(t), x, y) - f^{\text{avg}}_{\mathcal{S}_h}(\boldsymbol{a}(t))\right)^2 dx\, dy. \quad (7)$$

The OTPP involves finding a trajectory $\boldsymbol{a}(t)$ that minimizes the variation in film thickness defined by Eq. (7).

B. *Constraints for the OTPP*

In practice, there are constraints on the set of trajectories that are feasible because of constraints associated with realistic robotic manipulators. In particular, there are limits associated with the velocities and/or accelerations that can be developed to move the applicator along a reachable path.

In addition to the constraints imposed by the robotic manipulator, it may be desirable to actually further constrain the collection of feasible trajectories. For instance, it may be practical to consider only those trajectories where the applicator's positions are within a range of distance (e.g., between 8 and 12 inches) from the surface and/or consider only orientations where the centerline of the applicator's spray pattern is normal to the surface. Adding such intuitive constraints decreases the size of the search space and may improve the quality of the obtained solution and/or increase the possibility of attaining a globally optimal solution.

Two classes of constraint sets for the assumed feasible trajectories are defined in the following two subsections. In the first, the desired spatial path for the trajectory is assumed to be specified. In the second, the spatial constraints for the trajectories are relaxed and thus the set of feasible trajectories is represented by a collection of six-dimensional vector functions of time.

C. *The OTPP Along a Specified Spatial Path*

A spatial point for an applicator is characterized by a six-dimensional vector that defines the position and orientation of the applicator. The spatial path associated with an applicator trajectory is defined as the set of spatial points traversed by the applicator. In some applications, a desired spatial path for the applicator may be specified. In such cases, the question of interest is how to best traverse the specified spatial path as a function of time.

Let $p(\rho)$ denote a six-dimensional vector function that parameterizes a spatial path for the applicator. The scalar variable $\rho \in [0,1]$ parameterizes all points along the spatial path. Assume that the parameterization is such that $p(\rho)$ is a continuous function of the parameter ρ.

Consider a scalar function of time $\lambda(t)$, where $\lambda : [0,T] \rightarrow [0,1]$. By replacing the scalar parameter ρ with the scalar function $\lambda(t)$, the resulting vector functional $p(\lambda(t))$ is a characterization of trajectories that have spatial points along the spatial path $p(\rho)$. For practical reasons, it is generally necessary to constrain $\lambda(t)$ to be a continuous function in order to prevent discontinuous movements of the applicator (recall that $p(\rho)$ is also assumed to be continuous in ρ). It may further be necessary to limit the values of the first and/or second derivatives of $\lambda(t)$ in order to constrain the speed and/or acceleration of the applicator. Finally, constraining $\lambda(t)$ to be monotone increasing is tantamount to considering only those trajectories that do not backtrack along the specified spatial path. For notational convenience, the incorporation of all such desired constraints on $\lambda(t)$ are assumed to be included in a set of scalar functions denoted by $\Lambda(t)$.

Thus, the OTPP along a spatially parameterized path $p(\rho)$ is an optimization problem of the form

$$\min_{\lambda(t)\in\Lambda(t)} \{V_{S_h}(p(\lambda(t)))\}. \tag{8}$$

The search space for the above optimization problem, i.e., $\Lambda(t)$, is a set of scalar functions of time.

D. *The OTPP With General Constraints*

In the general case, the feasible constraint set can include trajectories that do not share the same spatial path. Let $\mathcal{A}(t)$ denote a general set of feasible applicator trajectories. Thus, $\mathcal{A}(t)$ is a set of six-dimensional vector functions of time. In practice, the union of all spatial points associated with all vector functions in $\mathcal{A}(t)$ is constrained by the region that is reachable by the robotic manipulator. Also, the translational and/or rotational velocities of the trajectories in $\mathcal{A}(t)$ may be constrained.

Therefore, the OTPP with general constraints is an optimization problem of the form

$$\min_{a(t)\in\mathcal{A}(t)} \{V_{S_h}(a(t))\}. \tag{9}$$

In contrast to the OTPP along a specified spatial path (where the search space is over a set of scalar functions of time) the optimization problem defined in Eq. (9) generically requires a search over a set of six-dimensional vector functions of time.

IV. Solution Techniques for OTPPs

The OTPP belongs to a general class of optimization problems known as constrained variational problems [6]. Constrained variational problems are analogous to, but more general than, problems that involve optimizing an ordinary function of several variables with constraints. One practical way to determine solutions to variational problems is to employ a so called direct method, which involves formulating the variational problem as a limit problem for some problem of extrema of a function of a finite number of variables [6]. The techniques described here for solving the OTPP are examples of direct methods.

A. *Solving the OTPP Along a Specified Spatial Path*

The proposed technique for solving the optimization problem of Eq. (8) is based on approximating $\lambda(t)$ as a piecewise constant function. Divide the interval $[0,T]$ into N subintervals, each of width $\Delta = T/N$. Let $b_k(t)$ denote a "boxcar" function, which is defined for each $k \in [1,2,\ldots,N]$ as follows:

$$b_k(t) = \begin{cases} 1 & \text{if } t \in [(k-1)\Delta, k\Delta] \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

Thus, a piecewise constant approximation for $\lambda(t)$ is given by

$$\tilde{\lambda}(t) = \sum_{k=1}^{N} \lambda_k b_k(t), \tag{11}$$

where $\lambda_k \in [0,1]$ represents the constant value of $\tilde{\lambda}(t)$ over the subinterval $[(k-1)\Delta, k\Delta]$.

By replacing $a(t)$ with $p(\tilde{\lambda}(t))$ in the right side of Eq. (3), it is straightforward to verify the following expression for film thickness

$$\tilde{f}_{S_h}(\boldsymbol{\lambda}, x, y) = \Delta \sum_{k=1}^{N} \dot{f}_{S_h}(p(\lambda_k), x, y), \tag{12}$$

where $\boldsymbol{\lambda} = [\lambda_1 \;\; \lambda_2 \;\; \cdots \;\; \lambda_N]'$.

Likewise, the corresponding expression for the variation in film thickness is given by

$$\tilde{V}_{S_h}(\boldsymbol{\lambda}) = \frac{1}{A_{S_h}} \int\!\!\int_{\mathcal{D}} \left(\tilde{f}_{S_h}(\boldsymbol{\lambda}, x, y) - \tilde{f}_{S_h}^{\mathrm{avg}}(\boldsymbol{\lambda})\right)^2 dx\, dy, \tag{13}$$

where

$$\tilde{f}^{\text{avg}}_{\mathcal{S}_h}(\boldsymbol{\lambda}) = \frac{1}{A_{\mathcal{S}_h}} \int\int_{\mathcal{D}} \tilde{f}_{\mathcal{S}_h}(\boldsymbol{\lambda}, x, y)\, dx\, dy. \qquad (14)$$

The function $\widetilde{V}_{\mathcal{S}_h}(\boldsymbol{\lambda})$ represents an approximation to the functional objective associated with the original variational problem of Eq. (8). Of course the constraints on the allowable set of functions from the original variational problem, i.e., $\Lambda(t)$, must also be transformed into a suitable constraint set for the vector $\boldsymbol{\lambda}$. Clearly, the range of values for λ_k is bounded by $0 \leq \lambda_k \leq 1$, for all $k \in [1, 2, \ldots, N]$. Also, in order to limit the speed among the feasible applicator trajectories, constraints may be placed on $|\lambda_k - \lambda_{k+1}|$, for each $k \in [1, 2, \ldots, N-1]$. Denote the set of vectors that include all such constraints on the vector $\boldsymbol{\lambda}$ by $\boldsymbol{\Lambda}$.

Thus, the nonlinear programming approximation to the OTPP along a spatially parameterized path $\boldsymbol{p}(\rho)$ has the form

$$\min_{\boldsymbol{\lambda} \in \boldsymbol{\Lambda}} \{\widetilde{V}_{\mathcal{S}_h}(\boldsymbol{\lambda})\}. \qquad (15)$$

Provided that the function $\widetilde{V}_{\mathcal{S}_h}(\boldsymbol{\lambda})$ is differentiable, then standard nonlinear programming techniques (e.g., gradient descent algorithms) can be employed to provide solutions to the optimization problem stated in Eq. (15) [4]. If the gradient of the objective function, i.e., $\nabla \widetilde{V}_{\mathcal{S}_h}(\boldsymbol{\lambda})$, can not be expressed analytically, then it can be approximated numerically. An instance of this problem is solved in Section V by employing the quasi-Newton method with a finite-difference approximation for the gradient.

B. *Solving the OTPP With General Constraints*

Analogous to the technique of the previous subsection, the technique proposed for solving the optimization problem of Eq. (9) is based on approximating $\boldsymbol{a}(t)$ as a piecewise constant vector function. Due to the space limitation, the detailed derivation of how to solve the OTPP with general constraints is included in [2].

V. Numerical Results

The optimization technique proposed in Section IV.A (for traversing a specified spatial path) is evaluated here for a particular example problem.

A. *Problem Setup*

The surface to be painted is a square flat panel that is located within the XY plane. The four corners of the panel are positioned at the XY coordinates $(1\frac{1}{3}, 0)$, $(1\frac{1}{3}, 5\frac{1}{3})$, $(6\frac{2}{3}, 5\frac{1}{3})$, and $(6\frac{2}{3}, 0)$. Using the notation of Eq. (1), the surface of the panel is denoted by $\mathcal{S}_0 = \{(x, y, 0) : 1\frac{1}{3} \leq x \leq 6\frac{2}{3} \;\&\; 0 \leq y \leq 5\frac{1}{3}\}$.

It is assumed that the applicator can be positioned above the panel so that the centerline of the spray pattern is oriented normal to the panel's surface. The orientation of applicator relative to the surface is assumed fixed. Letting the distance of the applicator from the surface be unity, the trajectory for the applicator has the form

$$\boldsymbol{a}(t) = [a_x(t) \;\; a_y(t) \;\; 1 \;\; 0 \;\; 0 \;\; 0]'. \qquad (16)$$

For each surface point $(x, y, 0) \in \mathcal{S}_0$, the assumed rate of film rate accumulation is given by $\dot{f}_{\mathcal{S}_0}(\boldsymbol{a}(t), x, y, t) =$

$$\frac{1}{[1 + (x - a_x(t))^2][1 + (y - a_y(t))^2]}. \qquad (17)$$

From Eq. (17), note that the maximum rate of film accumulation occurs at the XY coordinate where $x = a_x(t)$ and $y = a_y(t)$, i.e., at the XY coordinate directly under the applicator. This is consistent with the characteristics of many realistic applicators, which often have a "bell-shaped" distribution for the density of paint particles [11].

For convenience, the units of length are not specified here. In practice, the units for the dimensions of the panel may be on the order of a few feet or meters and the units for the rate of film accumulation could be on the order of a μm/sec.

Fig. 1 shows the XY coordinates of a parameterized spatial path denoted by $\boldsymbol{p}_{\ell,d}(\rho)$. The value of ℓ is the length of each straight segment associated with the four horizontal "sweeps" over the panel and d is the indexing distance between consecutive sweeps. The end-points of the horizontal segments are connected by semicircular arcs of radius $\frac{d}{2}$. An analytical parameterization of the path of the form $\boldsymbol{p}_{\ell,d}(\rho) = [p_x(\rho) \;\; p_y(\rho) \;\; 1 \;\; 0 \;\; 0 \;\; 0]'$, $\rho \in [0, 1]$, is given in [2].

B. *Traversing the Assumed Spatial Path at a Constant Speed*

The type of parameterization given in [2] for the path $\boldsymbol{p}_{\ell,d}(\rho)$ is known as a parameterization by arc length, which means that a unit change in the parameterizing variable ρ results in a unit change in curve length along the path [9]. Thus, trajectories of the form $\boldsymbol{p}_{\ell,d}(\frac{1}{T}t)$, for $t \in [0, T]$, represent a constant speed traversal of the spatial path $\boldsymbol{p}_{\ell,d}(\rho)$ over the time interval $[0, T]$.

Let $\tilde{\lambda}^{\text{lin}}_{T,N}(t)$ denote a piecewise constant approximation to a linearly increasing function of time, which is defined for each $t \in [0,T]$ by

$$\tilde{\lambda}^{\text{lin}}_{T,N}(t) = \sum_{k=1}^{N} \left(k - \frac{0.5}{N}\right) \Delta b_k(t), \qquad (18)$$

where $\Delta = T/N$ and the $b_k(t)$ is the boxcar function as defined by Eq. (10).

Fig. 2 shows the XY coordinates of the trajectory defined by $p_{\ell,d}(\tilde{\lambda}^{\text{lin}}_{T,N}(t)) + [1\frac{1}{3} \;\; 0 \; \cdots \; 0]'$ for the case $\ell = 5\frac{1}{3}$, $d = 1\frac{7}{9}$, $T = 8.34$, and $N = 74$. The 74 evenly spaced "$*$" symbols indicate the applicator's position during consecutive time intervals of width $\Delta = T/N = 0.1127$. Fig. 3 shows a contour plot for the panel's film thickness, which is a result of the constant speed trajectory.

C. Optimal Traversal of a Specified Spatial Path

The optimization technique of Section IV.A is applied to the spatial path $p_{\ell,d}(\rho) + [1\frac{1}{3} \;\; 0 \; \cdots \; 0]'$ with $\ell = 5\frac{1}{3}$ and $d = 1\frac{7}{9}$ (same as the spatial path assumed in Fig. 2). The values $N = 74$ and $T = 9.37$ are used to define the piecewise constant function $\tilde{\lambda}(t)$ of Eq. (11). The nonlinear function to be minimized, denoted below as $\tilde{V}_{S_0}(\boldsymbol{\lambda})$, is derived in [2]. An analytic expression for $\tilde{V}_{S_0}(\boldsymbol{\lambda})$ was possible for this example because the integral (over a flat surface) of the assumed formula for the rate of film accumulation, see Eq. (17), could be evaluated analytically.

$$\begin{aligned}
\tilde{V}_{S_0}(\boldsymbol{\lambda}) = {} & \\
& \frac{\Delta^2}{A_{S_0}} \left(\sum_{k=1}^{N} \left(\frac{x - p_{xk}}{2 + 2(x - p_{xk})^2} + \frac{1}{2}\tan^{-1}(x - p_{xk}) \right) \Bigg|_{\underline{x}}^{\overline{x}} \right. \\
& \qquad \left(\frac{y - p_{yk}}{2 + 2(y - p_{yk})^2} + \frac{1}{2}\tan^{-1}(y - p_{yk}) \right) \Bigg|_{\underline{y}}^{\overline{y}} \\
& \quad + 2 \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \left(\frac{\text{num}(x,i,j) - \text{num}(x,j,i)}{\text{den}(x,i,j)} \right) \Bigg|_{\underline{x}}^{\overline{x}} \\
& \qquad \left. \left(\frac{\text{num}(y,i,j) - \text{num}(y,j,i)}{\text{den}(y,i,j)} \right) \Bigg|_{\underline{x}}^{\overline{x}} \right) \\
& - \left(\frac{\Delta}{A_{S_0}} \sum_{k=1}^{N} \left(\tan^{-1}(x - p_{xk})\right)\Big|_{\underline{x}}^{\overline{x}} \left(\tan^{-1}(y - p_{yk})\right)\Big|_{\underline{y}}^{\overline{y}} \right)^2
\end{aligned} \qquad (19)$$

where

$$\begin{aligned}
\text{num}(x,i,j) &= -\ln(1 + x^2 - 2xp_{xi} + p_{xi}^2) \\
&\quad -(p_{xi} - p_{xj})\tan^{-1}(p_{xi} - x), \\
\text{den}(x,i,j) &= 4p_{xi} - 4p_{xj} - p_{xj}^3 \\
&\quad +3p_{xi}p_{xj}^2 - 3p_{xj}p_{xi}^2 + p_{xi}^3,
\end{aligned}$$

and p_{xk} and p_{yk} denote $p_x(\lambda_k)$ and $p_y(\lambda_k)$, respectively; the four corners of the panel have XY coordinates $(\underline{x}, \underline{y})$, $(\underline{x}, \overline{y})$, $(\overline{x}, \underline{y})$, and $(\overline{x}, \overline{y})$; the area of the flat surface is given by $A_{S_0} = (\overline{x} - \underline{x})(\overline{y} - \underline{y})$.

The assumed constraint set for $\boldsymbol{\lambda}$ is defined by

$$\boldsymbol{\Lambda} = \{\boldsymbol{\lambda} = [\lambda_1 \cdots \lambda_N]' : 0 \le \lambda_i \le 1\}. \qquad (20)$$

The nonlinear optimization problem of minimizing the function $\tilde{V}_{S_0}(\boldsymbol{\lambda})$ of Eq. (19) subject to the constraint set $\boldsymbol{\Lambda}$ of Eq. (20) was solved using an IMSL subroutine called BCONF, which uses a quasi-Newton method and a finite-difference gradient to minimize nonlinear functions with simple bounds on the variables [10]. The vector $\boldsymbol{\lambda}^{\text{lin}}_{T,N}$ with $T = 9.37$ and $N = 74$ was used as the initial condition for the algorithm. The default convergence parameters were used for the BCONF subroutine and the solution was obtained after about one hour of cpu time on a Sun Sparcstation 1. Fig. 4 shows the positions of the applicator based on the optimal solution $\tilde{\lambda}^*(t)$, which has an assumed time step $\Delta = T/N = 0.1266$. Fig. 5 shows a contour plot for the panel's film thickness, which is a result of the optimal trajectory.

Table I compares the performance features of three trajectories associated with two distinct spatial paths (constant and optimal speed traversal along the shorter of the two spatial paths and constant speed traversal along the longer spatial path). All trajectories result in an average film thickness of unity. The optimal trajectory (along the shorter spatial path) delivers a mean squared error that is about five times smaller than the constant speed traversal along the same spatial path. The total painting time (which is proportional to the amount of expended paint) for this optimal trajectory is about 12% more than the constant speed traversal of the same (short) spatial path and about 13% less than the constant speed traversal along the longer spatial path.

VI. Conclusions

A framework for solving an optimal trajectory planning problem (OTPP) for spray coating was developed. The proposed methodology is general in the sense that no real limitations are placed on the spray coating system nor the surface to be coated. The methodology can utilize empirically-based data for the rate film accumulation at each surface point, which is assumed to depend on the position and orientation of the applicator. It was demonstrated through an example that standard commercially available nonlinear

TABLE I. SIMULATION RESULTS

Trajectory Speed	ℓ / N	d / T	$\tilde{f}_{S_0}^{\text{avg}}(\boldsymbol{\lambda})$	$\tilde{V}_{S_0}(\boldsymbol{\lambda})$
Constant (Figs. 2 & 3)	5	$1\frac{7}{9}$		
	74	8.34	1.00	0.008714
Optimal (Figs. 4 & 5)	5	$1\frac{7}{9}$		
	74	9.37	1.00	0.001692
Constant (No Figures)	8	$1\frac{7}{9}$		
	100	10.57	1.00	0.004591

programming algorithms can be applied to solve the formulated optimization problem.

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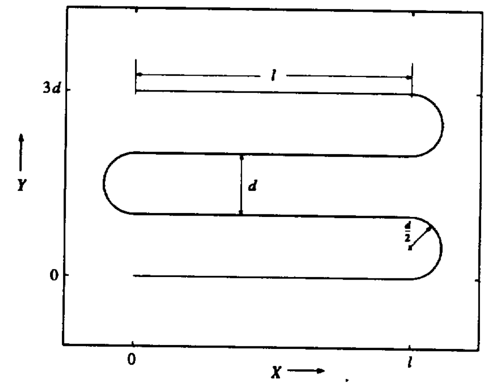

Fig. 1. The XY coordinates of the parameterized spatial path $\boldsymbol{p}_{\ell,d}(\rho)$.

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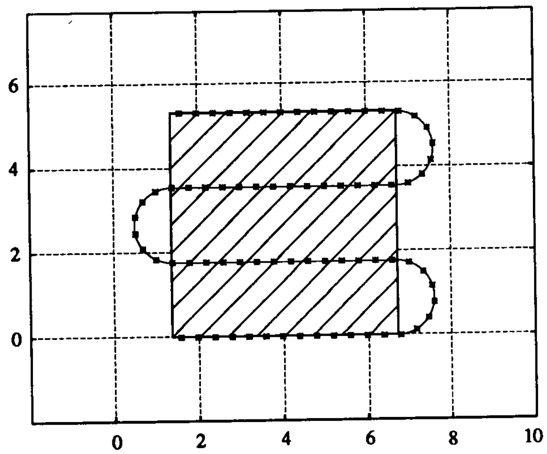

Fig.2. The XY coordinates of the trajectory $p_{\ell,d}(\tilde{\lambda}^{\text{lin}}_{T,N}(t))$ for the case $\ell = 5$, $d = 1\frac{7}{9}$, $T = 8.34$ and $N = 74$. The panel is indicated by the shaded area.

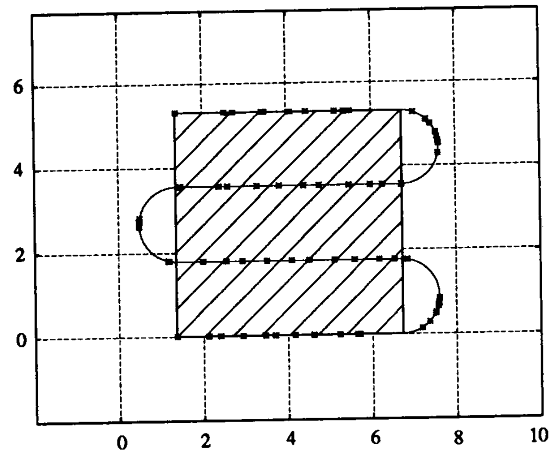

Fig. 4. The XY coordinates of the trajectory $p_{\ell,d}(\tilde{\lambda}^{*}_{T,N}(t))$ for the case $\ell = 5$, $d = 1\frac{7}{9}$, $T = 9.37$ and $N = 74$. The panel is indicated by the shaded area.

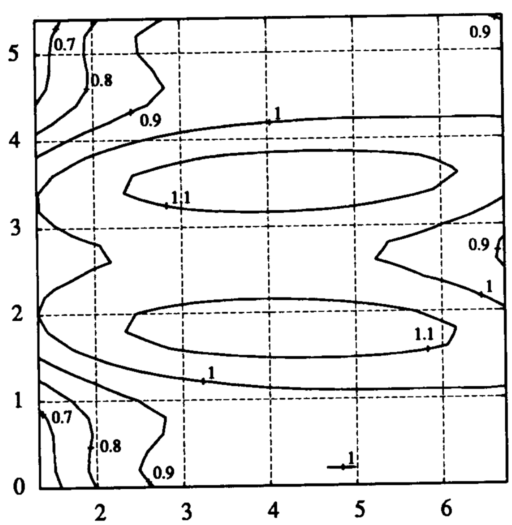

Fig. 3. The contour plot for the panel's film thickness that results from the trajectory of Fig. 2.

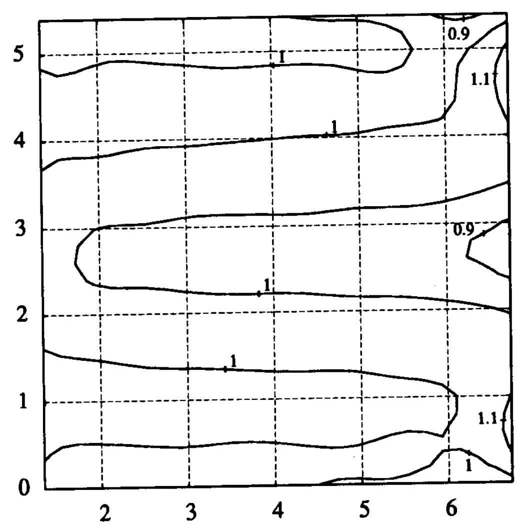

Fig. 5. The contour plot for the panel's film thickness that results from the trajectory of Fig. 4.