Complexity of Intensive Communications on Balanced Generalized Hypercubes*

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Abstract

Lower bound complexities are derived for three intensive communication patterns assuming a balanced generalized hypercube (BGHC) topology. The BGHC is a generalized hypercube [6] that has exactly w nodes along each of d dimensions for a total of w^d nodes. A BGHC is said to be dense if the w nodes along each dimension form a complete directed graph. A BGHC is said to be sparse if the w nodes along each dimension form a unidirectional ring. It is shown that a dense N node BGHC with a node degree equal to $K \log_2 N$, where $K \geq 2$, can process certain intensive communication patterns $K(K-1)$ times faster than an N node binary hypercube (which has a node degree equal to $\log_2 N$). Furthermore, a sparse N node BGHC with a node degree equal to $\frac{1}{L}\log_2 N$, where $L \geq 2$, is 2^L times slower at processing certain intensive communication patterns than an N node binary hypercube.

1 Introduction

The hypercube structure has been a popular choice for interconnecting large numbers of processing elements in parallel processing systems. Examples of commercially available parallel machines that utilize a hypercube interconnection network include nCUBE's nCUBE 2, Connection Machine's CM2, and Intel's iPSC2.

Unless stated otherwise, a d-dimensional hypercube is usually understood to mean a *binary* hypercube in which there are two connected nodes along each of d dimensions for a total of 2^d nodes. Some of the attractive features of the binary hypercube are: a relatively low number of links incident at each node (node degree $= d = \log_2 N$), a small minimum hop distance between nodes (network diameter $= d = \log_2 N$), and a large number of alternate paths between node pairs. Another very useful property of the hypercube is the fact that it is a symmetric graph.

In reference [6], Bhuyan and Agrawal introduce the concept of a generalized hypercube (GHC) in which w_i nodes are connected along the i^{th} dimension for a total of $N = \prod_{i=0}^{d-1} w_i$ nodes. One of the interesting properties of GHCs is the fact that for any given (integer) number of nodes N, there exists an integer $d \geq 1$ and a set of integers $\{w_0, w_1, \ldots, w_{d-1}\}$, $w_i \geq 2$, for which $N = \prod_{i=0}^{d-1} w_i$. If the number of nodes along any two dimensions are not equal (i.e., if $w_i \neq w_j$ for some i and j), then the resulting GHC will be an asymmetric graph. In the present paper, the complexities of three different intensive communication patterns are derived assuming a balanced GHC (BGHC) topology. In a BGHC, the number of nodes along every dimension equals w, i.e., $w_0 = w_1 = \cdots = w_{d-1} = w$.

A BGHC is said to be dense if the w nodes along each dimension are completely connected, which requires $(w-1)d$ input/output ports per node and a total of $(w-1)dw^d$ directed links. A BGHC is said to be sparse if the w nodes along each dimension are connected as a unidirectional ring, which requires d input/output ports per node and a total of dw^d directed links. Both the dense and sparse BGHCs are symmetric graphs. The dense interconnection pattern uses a maximal number of non-parallel directed links along each dimension; the sparse interconnection pattern maintains connectivity by using a minimal number of directed links along each dimension.

In references [1] and [2] communication patterns known as complete broadcast, single-node scatter, and total exchange are considered for binary hypercubes

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and optimal algorithms and complexities are derived. In the present paper, these same three patterns are considered and optimal complexities are derived assuming the interconnection network is either a dense BGHC or a sparse BGHC.[1] In the *complete broadcast pattern* each node distributes a local message to all other nodes, in the *single-node scatter pattern* a given node distributes distinct messages to each other node, and in the *total exchange pattern* each node distributes distinct messages to each other node.

It is assumed throughout this paper that all incident links of a node can be used to transmit and receive message simultaneously. This assumption is called *the multiple link availability assumption* in [1], *d-port communication* in [2], *the link bound model* in [4] and *the multiple acceptance scheme* in [5]. Furthermore, the following assumptions are made: message transmission is accomplished via the packet switching mode of communication, the buffer space at each node is infinite, and the time required to cross any link is the same for all links and is assumed to be one time unit.

The paper is organized as follows. In Section 2 some preliminary notation is introduced and the dense and sparse BGHCs are formally defined. In Section 3 the main complexity results are derived for the three communication patterns of interest (for both the dense and sparse BGHC topologies). The techniques used to derive lower bound complexities for dense BGHCs are relatively straightforward extensions of the techniques used in [1] for deriving complexity bounds for binary hypercubes. However, the derivations of lower bound complexities for sparse BGHCs are significantly different and more complicated than the case of dense BGHCs. Taken together, the complexity results for the dense and sparse BGHCs provide insightful performance criteria for a wide range of topological structures (i.e., ranging from rings to complete graphs). In Section 4, cost versus performance analysis between N node dense and N node sparse BGHCs is presented. The analysis indicates that: (1) an N node dense BGHC costing $K \geq 2$ times more than a corresponding N node binary hypercube can improve performance by a factor of $K(K-1)$ and (2) an N node sparse BGHC costing $L \geq 2$ times less than a corresponding N node binary hypercube degrades performance by a factor of 2^L.

[1] A BGHC with $w = 2$ is a binary hypercube, which is simultaneously dense and sparse.

2 The BGHC

The nodes in a BGHC are numbered 0 through $w^d - 1$ using a w-based numbering system. Node X is identified as $X = (x_{d-1}, x_{d-2}, \ldots, x_0)$, where $x_i \in \{0, 1, \ldots, w-1\}$ for each $i \in \{0, 1, \ldots, d-1\}$. For example, with $w = 4$ and $d = 2$, node 9 = (21). Following are definitions of some fundamental arithmetic operations.

Definition 1 *The* <u>*regular exclusive-or*</u> *of two coordinate values* $x, y \in \{0, 1, \ldots, w-1\}$ *is defined by*

$$x \oplus y = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{otherwise} \end{cases}$$

Definition 2 *The* <u>*w-modulated sum*</u> *of two coordinate values* $x, y \in \{0, 1, \ldots, w-1\}$ *is defined by*

$$x \oplus_w y = \begin{cases} x + y & \text{if } x + y \leq w - 1 \\ x + y - w & \text{if } x + y > w - 1 \end{cases}$$

Definition 3 *The* <u>*w-modulated difference*</u> *of two coordinate values* $x, y \in \{0, 1, \ldots, w-1\}$ *is defined by*

$$x \ominus_w y = \begin{cases} x - y & \text{if } x - y \geq 0 \\ x - y + w & \text{if } x - y < 0 \end{cases}$$

As a simple illustration of the above definitions, note that $6 \oplus 2 = 1$, $6 \oplus_7 3 = 2$, $6 \ominus_7 3 = 3$, and $3 \ominus_7 6 = 4$. The topologies associated with the dense and sparse BGHCs are formally defined next.

Definition 4
(a) *In a* <u>*dense BGHC*</u> *there is a directed arc from node* $X = (x_{d-1}, x_{d-2}, \ldots, x_0)$ *to node* $Y = (y_{d-1}, y_{d-2}, \ldots, y_0)$ *if and only if there is exactly one coordinate* i *for which* $x_i \oplus y_i = 1$. *We denote the coordinate system associated with this topology as* $\mathcal{K}_w^d$.
(b) *In a* <u>*sparse BGHC*</u> *there is a directed arc from node* $X = (x_{d-1}, x_{d-2}, \ldots, x_0)$ *to node* $Y = (y_{d-1}, y_{d-2}, \ldots, y_0)$ *if and only if there is exactly one coordinate* i *for which* $y_i = x_i \oplus_w 1$. *We denote the coordinate system associated with this topology as* $\mathcal{Z}_w^d$.

For the case of $w = 4$ and $d = 2$, the associated dense and sparse BGHCs are depicted in Fig. 1. An important measure associated with two given nodes, say X and Y, is the (minimum hop) distance from X to Y. As stated by Proposition 1 below, the distance between two nodes is determined by applying a simple modulated arithmetic operation to the w-based addresses of X and Y.

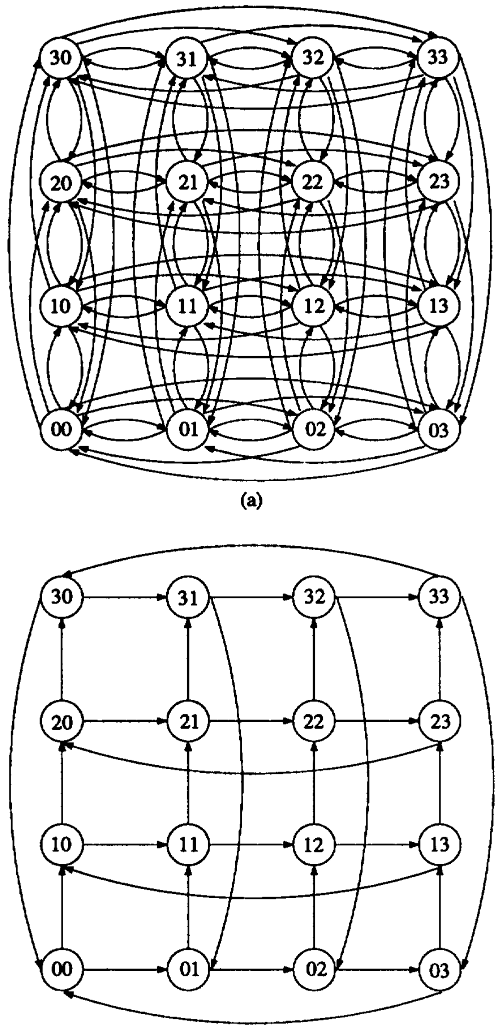

Figure 1: (a) The dense BGHC network with $w = 4$ and $d = 2$. The coordinate system for this network is denoted by $\mathcal{K}_4^2$. (b) The sparse BGHC network with $w = 4$ and $d = 2$. The coordinate system for this network is denoted by $\mathcal{Z}_4^2$.

Proposition 1
(a) *In the $\mathcal{K}_w^d$ coordinate system (the dense BGHC) the distance from X to Y is given by*

$$D_{\mathcal{K}_w^d}(X,Y) = \sum_{i=0}^{d-1} y_i \oplus x_i$$

(b) *In the $\mathcal{Z}_w^d$ coordinate system (the sparse BGHC) the distance from X to Y is given by*

$$D_{\mathcal{Z}_w^d}(X,Y) = \sum_{i=0}^{d-1} y_i \ominus_w x_i$$

Optimal broadcasting and scattering algorithms rely on the construction of carefully defined spanning trees rooted at certain nodes in the network. Fortunately, due to the symmetry of both the dense and sparse BGHCs, it suffices to simply consider spanning trees rooted at the origin node, say $O = (00\cdots0)$. In particular, given a spanning tree rooted at node O, the corresponding spanning tree rooted at an arbitrary node $X = (x_{d-1}, x_{d-2}, \ldots, x_0)$ is determined by performing a coordinate-wise w-modulated sum between the node addresses of the tree rooted at O and the address of node X. Fig. 2 shows an example of how a tree rooted at node $O = (00)$ in the $\mathcal{Z}_4^2$ coordinate system is converted to a tree rooted at node $X = (12)$ in the same coordinate system.

In defining spanning trees, it is important to classify all nodes that are of the same distance from the root node. The following definitions characterize the set of nodes that are a distance i from the node $O = (00\cdots0)$. Again, due to the symmetry of the BGHCs, the set of nodes that are a distance i from an arbitrary node X are obtained by performing coordinate-wise w-modulated sums between members of each set and the address associated with node X.

Definition 5
(a) *In the $\mathcal{K}_w^d$ coordinate system, let $\mathcal{D}^i_{\mathcal{K}_w^d}$ denote the set of nodes that are a distance i from the origin node, i.e.,*

$$\mathcal{D}^i_{\mathcal{K}_w^d} = \{X \mid D_{\mathcal{K}_w^d}(O,X) = i\}.$$

(b) *In the $\mathcal{Z}_w^d$ coordinate system, let $\mathcal{D}^i_{\mathcal{Z}_w^d}$ denote the set of nodes that are a distance i from the origin node, i.e.,*

$$\mathcal{D}^i_{\mathcal{Z}_w^d} = \{X \mid D_{\mathcal{Z}_w^d}(O,X) = i\}.$$

Proposition 2
(a) *In the $\mathcal{K}_w^d$ coordinate system, $\mathcal{D}^i_{\mathcal{K}_w^d} \neq \emptyset$, for all $i \in [0,1,\ldots,d]$ and $\mathcal{D}^i_{\mathcal{K}_w^d} = \emptyset$, for all $i \notin [0,1,\ldots,d]$.*

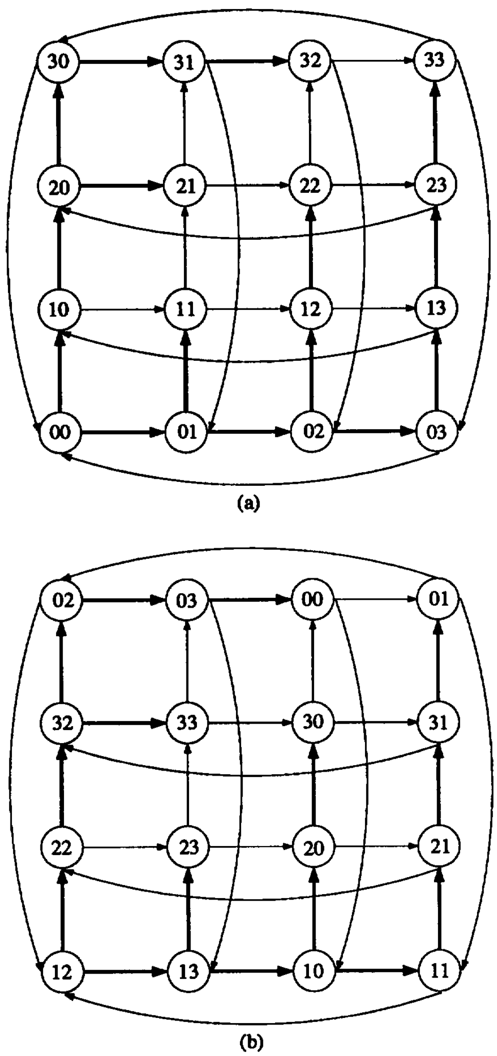

Figure 2: (a) A spanning tree rooted at node (00) on the sparse BGHC with $w = 2$ and $d = 2$. (b) The spanning tree of part (a) rooted at node (12). This tree is gotten by performing a coordinate-wise w-modulated sum between each node address and the address (12).

Furthermore, the diameter of the dense BGHC is equal to d.

(b) *In the $\mathcal{Z}_w^d$ coordinate system, $\mathcal{D}^i_{\mathcal{Z}_w^d} \neq \emptyset$, for all $i \in [0, 1, \ldots, (w-1)d]$ and $\mathcal{D}^i_{\mathcal{Z}_w^d} = \emptyset$, for all $i \notin [0, 1, \ldots, (w-1)d]$. Furthermore, the diameter of the sparse BGHC is equal to $(w-1)d$.*

3 Intensive Communications on BGHCs

Lower bounds for the complexities associated with intensive communication algorithms for both the dense and sparse BGHC topologies are derived in this section. For a given communication pattern and a given topology, the optimal time and transmission complexities are derived. The results for the case of dense and sparse BGHCs are summarized below in Tables 1 and 2, respectively (CB = Complete Broadcast, SNS = Single-Node Scatter, and TE = Total Exchange). The derivation of each entry in each table is presented in Subsections 3.1 and 3.2. In tabulating the results, all occurrences of w^d are expressed as N and all occurrences of d are expressed as $\frac{\log_2 N}{\log_2 w}$. The proofs of all required lemmas and theorems are in the Appendix. For the sake of comparison, Table 3 shows the complexities for the case of an N node binary hypercube. Table 1 and Table 2 (both) reduce to Table 3 by setting $w = 2$.

In Table 1 note that the complexities decrease as w is increased. This is to be expected because a dense BGHC becomes more dense as w is increased (i.e., more bandwidth is available as w is increased). In contrast, in Table 2 note that the complexities increase as w is increased. This is to be expected because a sparse BGHC becomes more sparse as w is increased (i.e., less bandwidth is available as w is increased).

Table 1: Optimal time and transmission complexities for an N node dense BGHC, $w \geq 2$.

Pattern	Time	Transmission
CB	$\lceil \frac{(N-1)\log_2 w}{(w-1)\log_2 N} \rceil$	$N(N-1)$
SNS	$\lceil \frac{(N-1)\log_2 w}{(w-1)\log_2 N} \rceil$	$\frac{N \log_2 N}{w \log_2 w}$
TE	$\lceil \frac{N}{w(w-1)} \rceil$	$\frac{N^2 \log_2 N}{w \log_2 w}$

Table 2: Optimal time and transmission complexities for an N node sparse BGHC, $w \geq 2$.

Pattern	Time	Transmission
CB	$\lceil\frac{(N-1)\log_2 w}{\log_2 N}\rceil$	$N(N-1)$
SNS	$\lceil\frac{(N-1)\log_2 w}{\log_2 N}\rceil$	$\frac{(w-1)N\log_2 N}{2\log_2 w}$
TE	$\lceil\frac{(w-1)N}{2}\rceil$	$\frac{(w-1)N^2\log_2 N}{2\log_2 w}$

Table 3: Optimal time and transmission complexities for an N node binary hypercube.

Pattern	Time	Transmission
CB	$\lceil\frac{N-1}{\log_2 N}\rceil$	$N(N-1)$
SNS	$\lceil\frac{N-1}{\log_2 N}\rceil$	$\frac{N\log_2 N}{2}$
TE	$\lceil\frac{N}{2}\rceil$	$\frac{N^2\log_2 N}{2}$

3.1 Intensive Communications on the Dense BGHC

3.1.1 Complete Broadcast on the Dense BGHC

Consider for a moment the simple single-node broadcast pattern in which a given node needs to distribute a single message to all other nodes. This pattern obviously requires at least $w^d - 1$ transmissions. Now, for the complete broadcast pattern, at least $w^d(w^d-1)$ transmissions are required because in a complete broadcast there are w^d single-node broadcasts occurring at once. In the dense BGHC recall that there are $(w-1)dw^d$ directed communication links. Therefore, if all of the communication resources are fully utilized at each time step, then the total time required to complete all required transmissions is bounded below by $\left\lceil\frac{w^d(w^d-1)}{(w-1)dw^d}\right\rceil = \left\lceil\frac{w^d-1}{(w-1)d}\right\rceil$.

3.1.2 Single-Node Scatter on the Dense BGHC

In the single-node scatter pattern, a given node must send $w^d - 1$ distinct packets successively through $(w-1)d$ incident links. Therefore, the amount of time required to send all packets out the node is bounded below by $\left\lceil\frac{w^d-1}{(w-1)d}\right\rceil$, which is a time lower bound for the single-node scatter pattern.

In order to derive a bound for the transmission complexity of the single-node scatter pattern, the number of nodes that are a distance i away from the given node must be known for each $i \in [0,1,\ldots,d]$. The following lemma provides the needed result.

Lemma 1 *In the $\mathcal{K}_w^d$ coordinate system, the following holds:*

$$\left|\mathcal{D}^i_{\mathcal{K}_w^d}\right| = \binom{d}{i}(w-1)^i, \qquad i \in [0,1\ldots,d].$$

Note that the summation over all i of $\left|\mathcal{D}^i_{\mathcal{K}_w^d}\right|$ should equal to the total number of nodes, namely w^d. Thus, as a simple check of Lemma 1:

$$\sum_{i=0}^{d}\left|\mathcal{D}^i_{\mathcal{K}_w^d}\right| = \sum_{i=0}^{d}\binom{d}{i}(w-1)^i$$

$$= \sum_{i=0}^{d}\binom{d}{i}(w-1)^i(1)^{d-i} = (w-1+1)^d = w^d,$$

where the third equality is from the binomial theorem.

A lower bound for the number of transmissions needed for the single node scatter pattern is proven by the next theorem.

Theorem 1 *The number of transmissions needed to process the single-node scatter pattern in the $\mathcal{K}_w^d$ coordinate system is bounded below by dw^{d-1}.*

3.1.3 Total Exchange on the Dense BGHC

The total exchange pattern is equivalent to w^d versions of the single-node scatter pattern taking place simultaneously. Therefore, the total number of transmissions required for the total exchange pattern is w^d times that of the single-node scatter pattern, or $w^d \cdot dw^{d-1} = dw^{2d-1}$. Now, if all of the $(w-1)dw^d$ links are used for transmission at each time unit, then $\left\lceil\frac{dw^{2d-1}}{(w-1)dw^d}\right\rceil = \left\lceil\frac{w^{d-1}}{(w-1)}\right\rceil$ time units are required for processing the total exchange pattern.

3.2 Intensive Communications on the Sparse BGHC

3.2.1 Complete Broadcast on the Sparse BGHC

In deriving a bound for the number of transmissions required for complete broadcast, only the total number of nodes in the network needs to be considered (i.e., the derivation is independent of the network topology). Thus, by following the same arguments stated in Subsection 3.1.1, a lower bound of $w^d(w^d-1)$ is obtained. To get a time lower bound, first note that the sparse BGHC has dw^d directed links. Therefore, if all of the communication resources are fully utilized at each time step, then the total time to complete all required transmissions is bounded below by $\left\lceil \frac{w^d(w^d-1)}{dw^d} \right\rceil = \left\lceil \frac{w^d-1}{d} \right\rceil$.

3.2.2 Single-Node Scatter on the Sparse BGHC

In the single-node scatter pattern, a given node must send $w^d - 1$ distinct packets successively through d incident links. Therefore, the amount of time required to send all packets out the node is bounded below by $\left\lceil \frac{w^d-1}{d} \right\rceil$, which is a time lower bound for the single-node scatter pattern.

In deriving a bound for the transmission complexity of the single-node scatter pattern, first consider the issue of determining the number of nodes which are of a distance i from the given node, where $i \in [0, 1, \ldots, (w-1)d]$. The following lemma provides a recursive equation for expressing $\left|\mathcal{D}^i_{\mathcal{Z}^d_w}\right|$.

Lemma 2 *In the $\mathcal{Z}^d_w$ coordinate system, the following holds:*

$$\left|\mathcal{D}^i_{\mathcal{Z}^d_w}\right| = \sum_{k=0}^{w-1} \left|\mathcal{D}^{i-k}_{\mathcal{Z}^{d-1}_w}\right|,$$

with initial conditions $\left|\mathcal{D}^i_{\mathcal{Z}^1_w}\right| = 1$ *for all* $i \in [0, 1, \ldots, (w-1)]$ *and* $\left|\mathcal{D}^i_{\mathcal{Z}^d_w}\right| = 0$ *for all* $i \notin [0, 1, \ldots, (w-1)d]$.

By assuming for a moment that $w = 2$ in Lemma 2, notice that the resulting recursion generates the well-known Pascal triangle. Thus, the recursion equation of Lemma 2 can be viewed as a natural generalization of the recursion associated with the Pascal triangle. Figure 3 shows example computations from the recursion for the case $w = 3$. Each nonzero value in each row of the "generalized" triangle correspond to values of $\left|\mathcal{D}^i_{\mathcal{Z}^d_3}\right|$, $i \in [0, 1, \ldots, 2d]$.

From Figure 3 it is clear that the sequence of numbers $\left\{\left|\mathcal{D}^i_{\mathcal{Z}^d_w}\right|\right\}_{i=0}^{(w-1)d}$ is symmetric. (Note: a sequence of numbers $\{c_i\}_{i=0}^n$ is said to be symmetric if and only if $c_i = c_{n-i}$, for all $i \in [0, 1, \ldots, \lfloor \frac{n}{2} \rfloor - 1]$.) Now, based on the fact that the sequence $\left\{\left|\mathcal{D}^i_{\mathcal{Z}^d_w}\right|\right\}_{i=0}^{(w-1)d}$ is symmetric and the fact that $\sum_{i=0}^{(w-1)d} \left|\mathcal{D}^i_{\mathcal{Z}^d_w}\right| = w^d$, a closed form expression for the sum $\sum_{i=0}^{(w-1)d} i \left|\mathcal{D}^i_{\mathcal{Z}^d_w}\right|$ is derived (which is a bound for the required number of transmissions), see Theorem 2. The key in deriving the closed form result is based on the the following lemma.

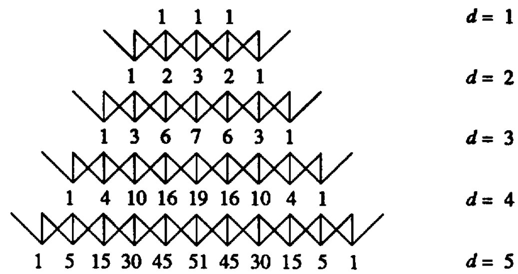

Figure 3: Example computations from the recursion equation of Lemma 2 with $w = 3$. For instance, note that $\left|\mathcal{D}^7_{\mathcal{Z}^5_3}\right| = 30$.

Lemma 3 *Given two symmetric sequences $\{a_i\}_{i=0}^n$ and $\{b_i\}_{i=0}^n$. If $\sum_{i=0}^n a_i = \sum_{i=0}^n b_i$, then $\sum_{i=0}^n i a_i = \sum_{i=0}^n i b_i$.*

Theorem 2 *The number of transmissions needed to process the single-node scatter pattern in the $\mathcal{Z}^d_w$ coordinate system is bounded below by $\frac{(w-1)dw^d}{2}$.*

3.2.3 Total Exchange on the Sparse BGHC

The total exchange pattern is equivalent to w^d versions of the single-node scatter pattern taking place simultaneously. Therefore, the total number of transmissions required for the total exchange pattern is w^d times that of the single-node scatter pattern, or $w^d \cdot \frac{(w-1)dw^d}{2} = \frac{(w-1)dw^{2d}}{2}$. Now, if all of the dw^d links are used for transmission at each time unit, then $\frac{(w-1)dw^{2d}}{2dw^d} = \frac{(w-1)w^d}{2}$ time units are required for processing the total exchange pattern.

4 Cost and Performance Analysis

Here, the cost of a network is defined as the number of directed links used. Thus, assuming N is a power of two, the cost of an N node binary hypercube is

$$C_{\text{binary}}(N) = N \log_2 N. \quad (1)$$

The cost of an N node dense BGHC with parameters w and d is $(w-1)dN$. Recalling that $N = w^d$, the cost of an N node dense BGHC can be expressed as a function of N and w:

$$C_{\text{dense}}(N, w) = \left(\frac{w-1}{\log_2 w}\right) N \log_2 N, \quad w \in [2, \ldots, N]. \tag{2}$$

Similarly, the cost of an N node sparse BGHC with parameters w and d is dN. Therefore, expressing this as a function of N and w:

$$C_{\text{sparse}}(N, w) = \left(\frac{1}{\log_2 w}\right) N \log_2 N, \quad w \in [2, \ldots, N]. \tag{3}$$

For a fixed number of nodes N, if the parameter w is increased, then the dense BGHC becomes more dense (i.e., more expensive), and the sparse BGHC becomes more sparse (i.e., less expensive). For the limiting case of $w = N$, the dense BGHC is a completely connected graph and the sparse BGHC is a unidirectional ring.

From Equations (1) and (2), note that an N node dense BGHC (with parameter $w \geq 2$) costs $\frac{w-1}{\log_2 w}$ times more than an N node binary hypercube. Also, from Tables 1 and 3 (of Section 3), note that the time complexity for performing the complete broadcast and the single-node scatter on a dense BGHC are (both) $\frac{w-1}{\log_2 w}$ times smaller than the corresponding complexities on a binary hypercube. Thus, with respect to these two patterns, increasing the cost of the underlying network by a factor of K improves the performance by a factor of K. However, for the total exchange pattern, note that the time complexity on the dense BGHC is $w(w-1)$ times smaller than that of the binary hypercube. Thus, with respect to the total exchange pattern, by increasing the cost of the underlying network by a factor of K, the resulting performance is improved by more than $K(K-1)$.

From Equations (1) and (3), note that an N node sparse BGHC (with parameter $w \geq 2$) costs $\log_2 w$ times less than an N node binary hypercube. Also, from Tables 2 and 3, note that the time complexity for performing the complete broadcast and the single-node scatter on a sparse BGHC are (both) $\log_2 w$ times larger than the corresponding complexities on a binary hypercube. Thus, with respect to these two patterns, decreasing the cost of the underlying network by a factor of L degrades the performance by a factor of L. However, for the total exchange pattern, note that the time complexity on the sparse BGHC is $(w-1)$ times larger than that of the binary hypercube. Thus, with respect to the total exchange pattern, by decreasing the cost of the underlying network by a factor of L, the resulting performance is degraded by a factor of 2^L.

5 Summary

Lower bound time and transmission complexity formulas were derived for three intensive communication patterns assuming two separate classes of generalized hypercubes. The two classes of generalized hypercubes are called dense BGHCs and sparse BGHCs. For a fixed number of nodes, the dense BGHC is more dense than the corresponding binary hypercube and the sparse BGHC is more sparse than the corresponding binary hypercube (given that all graphs have the same number of nodes and that $w > 2$). Cost and performance analysis shows that for processing the most intensive of the three communication patterns (i.e., the total exchange pattern), a slight cost penalty which may result by employing the dense BGHC (over the binary hypercube) is compensated by a substantial gain in performance. Also, it is shown that a slight decrease in cost resulting from using the sparse BGHC (instead of the binary hypercube) can degrade performance dramatically.

The practical insight from the paper is summarized as follows. In order to achieve reasonable cost performance ratios for extremely intensive communication patterns, the underlying network should be at least as dense as a binary hypercube.

Finally, we note that communication algorithms that achieve all of the lower bound complexities presented in this paper have been derived and will be submitted for publication in the near future.

Appendix

This appendix contains the proofs of all lemmas and theorems used to derive the complexity results of Section 3.

Lemma 1 *In the $\mathcal{K}_w^d$ coordinate system, the following holds:*

$$\left|\mathcal{D}^i_{\mathcal{K}_w^d}\right| = \binom{d}{i} (w-1)^i, \qquad i \in [0, 1 \ldots, d].$$

Proof: There are $\binom{d}{d-i} = \binom{d}{i}$ different ways to place $d - i$ zeros among d coordinate positions. Moreover, each of the remaining i nonzero coordinate positions can take on $(w-1)$ possible values.
Q.E.D.

Theorem 1 *The number of transmissions needed to process the single-node scatter pattern in the $\mathcal{K}_w^d$ coordinate system is bounded below by dw^{d-1}.*

Proof: Transmitting a distinct packet from the origin node to any given node in the set $\mathcal{D}^i_{\mathcal{K}^d_w}$ requires i transmissions. Now, because the sets $\mathcal{D}^i_{\mathcal{K}^d_w}$ partition the set of all nodes (taken over all i) and because a distinct packet must be sent from the origin to every other node, the total number of transmissions required is necessarily bounded below by

$$\sum_{i=0}^{d} i\left|\mathcal{D}^i_{\mathcal{K}^d_w}\right| = \sum_{i=1}^{d} i \binom{d}{i} (w-1)^i (1)^{d-i} = dw^{d-1}.$$

Q.E.D.

Lemma 2 *In the $\mathcal{Z}^d_w$ coordinate system, the following holds:*

$$\left|\mathcal{D}^i_{\mathcal{Z}^d_w}\right| = \sum_{k=0}^{w-1} \left|\mathcal{D}^{i-k}_{\mathcal{Z}^{d-1}_w}\right|,$$

with initial conditions $\left|\mathcal{D}^i_{\mathcal{Z}^1_w}\right| = 1$ *for all* $i \in [0,1,\ldots,(w-1)]$ *and* $\left|\mathcal{D}^i_{\mathcal{Z}^d_w}\right| = 0$ *for all* $i \notin [0,1,\ldots,(w-1)d]$.

Proof: First note that $\left|\mathcal{D}^i_{\mathcal{Z}^1_w}\right| = 1$ for all $i \in [0,1,\ldots,(w-1)]$ because $\mathcal{Z}^1_w$ is simply a unidirectional ring. Now, from Definition 5(b) and Proposition 1(b) the following alternate definition for $\mathcal{D}^i_{\mathcal{Z}^d_w}$ is apparent:

$$\mathcal{D}^i_{\mathcal{Z}^d_w} = \left\{ X = (x_{d-1}, x_{d-2}, \ldots, x_1, x_0) \mid \sum_{k=0}^{d-1} x_k = i \right\}.$$

So, the number of ways to choose d coordinates (having values in the range $[0,1,\ldots(w-1)]$) so that their sum is i can be expressed as the sum of the number of ways to sum $d-1$ coordinates so that their sum is i (i.e., choose $x_{d-1} = 0$) plus the number of ways to choose $d-1$ coordinates so that their sum is $i-1$ (i.e., choose $x_{d-1} = 1$), etc, until reaching $i - w + 1$ (because the the maximum possible value of x_{d-1} is $(w-1)$).
Q.E.D

Lemma 3 *Given two symmetric sequences $\{a_i\}_{i=0}^n$ and $\{b_i\}_{i=0}^n$. If $\sum_{i=0}^n a_i = \sum_{i=0}^n b_i$, then $\sum_{i=0}^n ia_i = \sum_{i=0}^n ib_i$.*

Proof: Define $S = \sum_{i=0}^n a_i = \sum_{i=0}^n b_i$.

If n is odd, then $\sum_{i=0}^n ia_i = \sum_{i=0}^{\lfloor\frac{n}{2}\rfloor-1} ia_i + (n-i)a_{n-i} = \sum_{i=0}^{\lfloor\frac{n}{2}\rfloor-1}(i+n-i)a_i = n\sum_{i=0}^{\lfloor\frac{n}{2}\rfloor-1} a_i = \frac{nS}{2}$. Likewise, $\sum_{i=0}^n ib_i = \frac{nS}{2}$.

If n is even, then $\sum_{i=0}^n ia_i = a_{\frac{n}{2}} + \sum_{i=0}^{\lfloor\frac{n}{2}\rfloor-1} ia_i + (n-i)a_{n-i} = a_{\frac{n}{2}} + \sum_{i=0}^{\lfloor\frac{n}{2}\rfloor-1}(i+n-i)a_i = a_{\frac{n}{2}} + n\sum_{i=0}^{\lfloor\frac{n}{2}\rfloor-1} a_i = a_{\frac{n}{2}} + \frac{nS}{2} - a_{\frac{n}{2}} = \frac{nS}{2}$. Likewise, $\sum_{i=0}^n ib_i = \frac{nS}{2}$.
Q.E.D

Theorem 2 *The number of transmissions needed to process the single-node scatter pattern in the $\mathcal{Z}^d_w$ coordinate system is bounded below by $\frac{(w-1)dw^d}{2}$.*

Proof: Define $a_i = \left|\mathcal{D}^i_{\mathcal{Z}^d_w}\right|$ and $b_i = \binom{(w-1)d}{i} \left(\frac{1}{2}\right)^{(w-1)d} w^d$, $i \in [0,1,\ldots,(w-1)d]$. By Lemma 2, the sequence $\{a_i\}_{i=0}^{(w-1)d}$ is symmetric and $\sum_{i=0}^{(w-1)d} a_i = w^d$. Likewise, because each b_i is simply a scaled binomial coefficient, note that the sequence $\{b_i\}_{i=0}^{(w-1)d}$ is symmetric and that $\sum_{i=0}^{(w-1)d} b_i = 2^{(w-1)d}\left(\frac{1}{2}\right)^{(w-1)d} w^d = w^d$. Finally, by applying Lemma 3, it is apparent that $\sum_{i=0}^{(w-1)d} ia_i = \sum_{i=0}^{(w-1)d} ib_i = \frac{(w-1)dw^d}{2}$.
Q.E.D.

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